Classifying simple fusion systems and groups

I'm going to engage in a highly speculative discussion of the following two problems:

Problem 1. Determine all simple saturated fusion systems at the prime p.

Problem 2. Use the classification of simple saturated fusion systems at the prime 2 to give a new proof of the classification of the finite simple groups.

I have no real reason to believe that it is possible to solve either problem. But what does seem possible is that some parts of the proof of the classification of the simple groups, might be easier to implement in the setting of fusion systems. If this is true of enough such steps, it might be possible to solve Problem 1 at the prime 2. Then if Problem 2 has a nice solution, one could obtain a simpler proof of the classification of simple groups.

In any event it is interesting to speculate about such matters, and that is what I will do today. Moreover in the process some concrete problems will emerge, problems it may well be possible to solve.

Normal subsystems, factor systems, and simple systems

Let p be a prime and $\mathcal{F}$ a saturated fusion system on a p-group S. A year and a half ago in Banff, I gave a definition of a *normal subsystem* of $\mathcal{F}$ on a subgroup T of S. I won't repeat that definition today, but you can find it and a discussion of such objects in "Notes on the local theory of fusion systems" on the web page for the workshop in Birmingham next week:

www.bham.ac.uk/C.W.Parker/Fusion/lmsfusion.htm

I recall however that T is forced to be strongly closed in S with respect to $\mathcal{F}$.

We can now define $\mathcal{F}$ to be *simple* if it has no nontrivial proper normal subsystems.

Let T be a subgroup strongly closed in S with respect to $\mathcal{F}$. In the notes there is also a definition of a factor system $\mathcal{F}/T$ on S/T, and a surjective morphism $\Theta : \mathcal{F} \to \mathcal{F}/T$. Moreover there is a bijection between strongly closed subgroups and homomorphic images of $\mathcal{F}$. It is not true that for each such T there is a normal subsystem of $\mathcal{F}$ on T. Further if $\mathcal{D} \trianglelefteq \mathcal{F}/T$, its preimage in $\mathcal{F}$ need not be normal in $\mathcal{F}$. Still the notions are close enough to the usual notions for groups that one can define the notion of a *composition series* for $\mathcal{F}$, and prove a Jordon-Holder Theorem.

Thus it makes sense to study simple systems.

Examples.

What are examples of simple systems? It is easy to see the Benson-Solomon systems are simple. Further since most of our examples of fusion systems come from groups, probably the first obvious question is: If $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G and $S \in Syl_p(G)$, what are the composition factors of $\mathcal{F}$? From the Jordon-Holder Theorem, the factors for $\mathcal{F}$ are the union of the composition factors for $\mathcal{F}_{S_i}(L_i)$, as L_i varies over the composition factors of G with $p \in \pi(L_i)$, and $S_i \in Syl_p(L_i)$. So we are reduced to:

Problem 3. Let G be simple, $S \in Syl_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. What are the composition factors of $\mathcal{F}$?

I've worked out the answer except when G is of Lie type in characteristic $r \neq p$, and either;

(a) $p = 2$, or

(b) p is odd and G is classical.

It turns out that in case (b), there can exist exotic composition factors. Ruiz has a complete treatment (phrased differently) in the case $G = L_n(q)$ and p odd. In any event we have:

Problem 4. Determine the composition factors of G in the exceptional cases (a) and (b).

About two weeks ago I received a paper from R. Flores and R.Foote, "Strongly closed subgroups and the cellular structure of classifying spaces", which determines all strongly closed subgroups in finite groups, and in particular in simple groups. This is one piece of information one needs for Problem 3.

Recognition Theorems

Suppose that we have compiled a list $\mathcal{K}$ of simple systems that we hope is complete. Given a simple system $\mathcal{F}$, how to we recognize $\mathcal{F}$? That is we want a set of local data which will allows us to prove that $\mathcal{F}$ is isomorphic to some member of $\mathcal{K}$. Here is one possible approach:

Given $\mathcal{F}$, suppose there exist $R_i \in \mathcal{F}^{frc}$ for $i = 1, 2$, such that

(1) $R_1 \trianglelefteq S$.

(2) $\mathcal{F} = \langle \mathcal{F}_1, \mathcal{F}_2 \rangle$, where $\mathcal{F}_i = N_{\mathcal{F}}(R_i)$.

(3) $Aut_{\mathcal{F}}(R_2) = \langle Aut_{N_{\mathcal{F}_2}(N_{R_1}(R_2))}(R_2), \phi \rangle$, for suitable $\phi \in Aut_{\mathcal{F}}(R_2)$.

(4) $N_{\mathcal{F}_2}(N_{R_1}(R_2)) = N_{\mathcal{F}_1}(R_2)$.

Suppose these hypotheses hold. Then by (1) and Proposition C in[BCGLO1]), there exists a model G_i for $\mathcal{F}_i$. That is G_i is a finite group with $N_S(R_i) = S_i \in Syl_p(G_i)$, $F^*(G_i) = O_p(G_i)$, and $\mathcal{F}_i = \mathcal{F}_{S_i}(G_i)$. Then $N_{\mathcal{F}_1}(R_2) = \mathcal{F}_{S_2}(G_{1,2})$, where $G_{1,2} = N_{G_1}(R_2))$, and $N_{\mathcal{F}_2}(N_{R_1}(R_2)) = \mathcal{F}_{S_2}(G_{2,1})$, where $G_{2,1} = N_{G_2}(N_{R_1}(R_2))$. Then by (4), there is an isomorphism $\alpha : G_{1,2} \to G_{2,1}$ extending the identity map on S_2, so we obtain an amalgam $\mathcal{A} = (G_1 \stackrel{\iota}{\leftarrow} G_{1,2} \stackrel{\alpha}{\rightarrow} G_2)$. Further by (2) and (3),

$$\mathcal{F} = \langle \mathcal{F}_1, \mathcal{F}_2 \rangle = \langle \mathcal{F}_S(G_1), \mathcal{F}_{S_2}(G_2) \rangle = \langle \mathcal{F}_S(G_1), c_{g|R_2} \rangle,$$

where $g \in G_2$ with $c_{g|R_2} = \phi$.

Now suppose $\tilde{\mathcal{F}}$ is a simple system on S such that $\tilde{\mathcal{F}}$ also satisfies (1)-(4) on the same subgroups R_1 and R_2. Form the amalgam $\tilde{\mathcal{A}}$ and suppose we can produce an isomorphism $\beta : \mathcal{A} \to \tilde{\mathcal{A}}$ extending the identity map on S. Then $\beta : \mathcal{F} \to \tilde{\mathcal{F}}$ is an isomorphism, as

$$\mathcal{F}\beta = \langle \mathcal{F}_S(G_1)\beta, c_{g|R_2}\beta \rangle = \langle \mathcal{F}_S(\tilde{G}_1), c_{g\beta|R_2} \rangle = \tilde{\mathcal{F}}.$$

Note: Recognition of fusion systems seems easier than recognition of groups.

Systems of even characteristic

Assume now $p = 2$. Define $\mathcal{F}$ to be of *characteristic 2-type* if for all $1 \neq R \in \mathcal{F}^f$, $N_{\mathcal{F}}(R)$ is constrained. Define $\mathcal{F}$ to be of *even characteristic* if for all $1 \neq R \trianglelefteq S$, $N_{\mathcal{F}}(R)$ is constrained.

Following either the original proof of the Classification, or the GLS revision (or perhaps more precisely Aschbacher-Smith [AS] on quasithin groups), we obtain two partitions of our simple systems; eg. In one partition, the blocks are the set of systems of even characteristic, and the set of those not of even characteristic.

In the case of the Classification, when one encounters a group of even characteristic, the strategy is to pass to the study of r-local subgroups for some suitable odd prime r. For fusion systems, we don't have that luxury. However one can hope to use some of the techniques of the Meierfrankenfeld et al program, or similar techniques in Aschbacher-Smith. Whether such techniques are strong enough remains an open question, which I will not consider here. Instead I'll spend the rest of my time on what Steve and Ron Solomon (essentially) call *GW-systems* (ie. Gorenstein-Walter systems) : the systems which are not of even characteristic).

Note that instead we could consider systems not of characteristic 2-type. But then we would have to deal with the systems $\mathcal{F}_S(G)$, where G is a wreath product of a group L of

Lie type and even characteristic, by $\mathbf{Z}_2$, or where G is L extended by an involutory field automorphism. If there is an easy way locally to see such systems are not simple, then it is perhaps easier to work with this partition, as the treatment of systems of characteristic 2-type is presumably easier than that of systems of even characteristic.

Quasisimple systems

So assume $\mathcal{F}$ is a simple GW-system. Then there is an involution $z \in Z(S)$ such that $C_{\mathcal{F}}(z)$ is not constrained. In my Banff talk and/or the Birmingham Notes, I defined the *Generalized Fitting subsystem* $F^*(\mathcal{E})$ of a saturated fusion system $\mathcal{E}$. Here $F^*(\mathcal{E}) = O_p(\mathcal{E})E(\mathcal{E})$, and $E(\mathcal{E})$ is the central product of the components of $\mathcal{E}$: the subnormal quasisimple subsystems. In particular $\mathcal{E}$ is constrained iff $E(\mathcal{E}) = 1$, so $C_{\mathcal{F}}(z)$ has a component $\mathcal{C}$. Proceeding by induction on the order of $\mathcal{F}$, we can assume $\mathcal{C}/Z(\mathcal{C}) \in \mathcal{K}$.

At this point we see that we need to know the *coverings* of simple systems $\mathcal{E} \in \mathcal{K}$; that is a covering $\pi : \mathcal{D} \to \mathcal{E}$ is a surjective morphism such that $Z(\mathcal{D})$ is the kernel of π and $\mathcal{D}$ is 2-perfect: $\mathcal{D} = O^2(\mathcal{D})$.

I have not worked out the details but it seems to me that by general nonsense (using a fiber product argument and the existence of direct products of fusion systems), that there is a universal covering $\tilde{\mathcal{E}}$. Moreover if $\mathcal{E} = \mathcal{F}_S(G)$ is the system of a group, then G is simple, and by Corollary 6.14 in [BCGLO2], $\tilde{\mathcal{E}} = \mathcal{F}_{\tilde{S}}(\tilde{G})$, where $\tilde{G}$ is the universal covering group of G.

My guess is the Benson-Solomon systems are simply connected. But in any event the covering systems of the simple exotic 2-systems would also need to be determined.

No Cores

In the treatment of GW-groups in the Classification, the cores $O(C_G(i))$ of involutions i are a serious obstacle. With great effort, one proves the B-conjecture via a treatment of the unbalanced groups. Then one proves G has a *standard component*. The major tools are the Gorenstein-Walter theorem on L-balance, and signalizer functor theory.

The big advantage of fusion systems is that they have no cores. We don't need to prove the B-conjecture,

Recall also that in Banff and/or the Birmingham notes, I indicated I have a proof of L-balance for fusion systems: For $U \in \mathcal{F}^f$, $E(C_{\mathcal{F}}(U)) \leq E(\mathcal{F})$.

Classical involutions and Walter's Theorem

Let G be a finite group. To oversimplify a bit, a *classical involution* in G is an involution z such that there exists $z \in K \trianglelefteq\trianglelefteq C_G(z)$ with $K \cong SL_2(q)$ for some odd prime power q. The Classical Involution Theorem says essentially that if G has a classical involution then G is of Lie type and odd characteristic.

Then Walter's Theorem essentially says if i is an involution in G such that $C_G(i)$ has a component of Lie type and odd characteristic, then G is of Lie type and odd characteristic. The proof involves producing a classical involution and appealing to the Classical Involution Theorem.

It seems possible that the same theorems could be proved for fusion systems, except because of No Cores, the proofs may well be much easier. In particular this would allow us to assume that components of centralizers of involutions in our simple fusion system, or not the systems of groups of Lie type and odd characteristic.

Specifically:

Hypothesis Ω. $\mathcal{F}$ is a saturated fusion system on a finite 2-group S, and Ω is a collection of subgroups of S such that $\Omega^{\mathcal{F}} = \Omega$ and

(1) There exists $e \geq 3$ such that for all $K \in \Omega$, K has a unique involution $z(K)$ and K is nonabelian of order 2^e.

(2) For each pair of distinct $K, J \in \Omega$, $|K \cap J| \leq 2$ with $[K, J] = 1$ in case of equality.

(3) If $K, J \in \Omega$ and $v \in J - Z(J)$, then $v^{\mathcal{F}} \cap C_S(z(K)) \subseteq N_S(K)$.

(4) If z is the involution in $K, J \in \Omega$, $v \in K$, and $\phi \in \hom_{C_{\mathcal{F}}(z)}(\langle v \rangle, S)$, then $v\phi \in J$ or $v\phi$ centralizes J.

Problem 5. Extend the various Theorems in [A4] to results about fusion systems satisfying Hypothesis Ω. In particular if $\mathcal{F}$ is simple, show that (essentially) $\mathcal{F}$ is the fusion system of some group of Lie type and odd characteristic.

Problem 6. Extend Theorem III in [W] to the domain of saturated fusion systems at the prime 2.

Namely consider the following hypothesis (or something like it):

Hypothesis W. $\mathcal{F}$ is a saturated fusion system on a finite 2-group S with $F^*(\mathcal{F})$ quasisimple. Assume there exists an involution $i \in S$ such that $\langle i \rangle \in \mathcal{F}^f$ and $C_{\mathcal{F}}(i)$ has a component in $Chev^*(r)$ for some odd prime r.

Here $Chev^*(r)$ is essentially the class of fusion systems of quasisimple groups of Lie type and odd characteristic, distinct from $L_2(r^e)$. One would like to show that if Hypothesis W holds, then, with known exceptions, Hypothesis Ω holds. Note that one class of exceptions are the exotic systems of Solomon and Benson, constructed by Levi and Oliver. These arise in [W] during the proof of Proposition 4.3 of that paper.

Problem 7. Extend the Component Theorem of [A3] to the domain of saturated fusion systems at the prime 2.

This extension would say that, modulo known exceptions, if $\mathcal{F}$ is a saturated fusion system on a finite 2-group S, and there exists an involution $i \in S$ such that $\langle i \rangle \in \mathcal{F}^f$ and $C_{\mathcal{F}}(i)$ is not constrained, then there exists a "standard component" in the centralizer of some involution.

If we work with groups of even characteristic, this is not quite the right result. Instead, one should also assume $i \in Z(S)$, and proceed as in Chapter 16 of [AS].

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