Tillman : $\boxed{K(n)\text{- compact groups}}$

§1. Some stable homotopy

Let K be any (co)homology theory and $X \longrightarrow L_K X$ be the unstable localization, $E \longrightarrow L_K E$ the stable localization.

Denote by Sp_K the category of K-local spectra with unit $L_K \mathbb{S}$ and smash product $E \wedge F = L_K(E \wedge F)$ and mapping spectra $map(E,F) = L_K\, map(E,F)$.

Dualizable objects

For any spectrum define $DE = map(E, L_K \mathbb{S})$

There is a natural map

$$DE \wedge E \longrightarrow map(E,E)$$

If this is an equivalence, we call E <u>dualizable</u>.

In this case $DDE \simeq E$.

Examples

1) $K = \mathbb{S}, H\mathbb{Z}$ then E is dualizable iff E is a finite CW-spectrum.

2) $K = H\mathbb{F}_p$, then E is dualizable iff $H_*(E, \mathbb{F}_p)$ is finite.

3) $K = K(n)$, then E is dualizable iff $K(n)_*(E)$ is a f. dim. $K(n)_*$-vector space

(in general this is not true). Recall that $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$ with $|v_n| = 2p^n - 2$, this is a graded field, so a module which respects the grading over $K(n)_*$ is free.

K_n is an extension of $K(n)$ which also has Künneth iso. $(K_n)_* = \mathbb{F}_{p^n}[u, u^{-1}]$, $|u| = 2$

Just like $\mathbb{F}_p$ is the residue field of $\mathbb{Z}_{\hat{p}}$, $K(n)_*$ is the residue field of $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}]$ and $(K_n)_*$ of $(E_n)_* = \underbrace{\mathbb{Z}_{p^n}}_{\text{Witt vectors } W(\mathbb{F}_{p^n})}[[v_1, \dots, v_{n-1}]][u, u^{-1}]$

§2. K(n)- compact groups

Def: A K- compact group is a topological group G (or a loop space) which is K- local and such that the suspension spectrum localized at K $L_K \Sigma^\infty_+ G$ is dualizable.

Note: If one would assume BG is K- local, one can not see whether p- compact group are K(n)-compact groups.

Example: If $K(n)_*(L_{K(n)}\Sigma^\infty_+ G)$ is a finite $K(n)_*$- vector space (so G is a K(n)-compact group), then this is equivalent to require that $K(n)_*(G)$ be finite. In this case

$$G \text{ } K(n)\text{- compact} \iff \begin{cases} G \text{ is } K(n)\text{ - local} \\ G \text{ is } K(n)\text{- finite.} \end{cases}$$

Examples

(1) $G = L_{K(n)}$ (p- compact group). We have to show that

$K(n)_*(G)$ is finite. Use the AHSS
$\underbrace{H_*(G; K(n)_*)}_{\text{finite } K(n)_*\text{-v.s.}} \Longrightarrow K(n)_*(G)$ must be finite.

If one would like to have BG local, one should take $L_{K(n)}BG$ and compute $\Omega L_{K(n)} BG$. It is not known if this is finite in K(n)- homology

$BS^1{}^\wedge_p = \mathbb{C}P^\infty{}^\wedge_p$ for example is $K(n)$-local, $n \geq 1$. So there is no problem, but for other BG's...

② <u>Abelian examples</u>: $K(n)_*(K(\mathbb{Z}/p^k, m)) = \begin{cases} \text{finite} & k \neq 0\ m \leq n \\ 0 & m > n \end{cases}$ (Ravenel–Wilson)

In fact Bousfield proved that $K(\mathbb{Z}/p^k, m)$ is either local or acyclic. Therefore all $K(\mathbb{Z}/p^k, m)$, $m \leq n$, are $K(n)$-compact groups.

Likewise Ravenel and Wilson computed

$K(n)_*(K(\hat{\mathbb{Z}}_p, m))$ is ∞ dimensional, unless $m > n+1$, where $K(\hat{\mathbb{Z}}_p, m)$ is acyclic, or $m = 1$. So the only $K(\hat{\mathbb{Z}}_p, m)$ which is a $K(n)$-compact group is $(S^1)^\wedge_p$!

③ Open problem: Is $\Omega L_{K(n)} BS^3$ $K(n)$-finite?

<u>§3. Dimension</u>

Let G be a K-compact group. ~~Define~~

$$S^{\mathfrak{g}} = (\Sigma^\infty_+ G)^{hG^{op}} = \mathrm{map}^G(EG_+, \Sigma^\infty_+ G).$$

This is a spectrum in Sp_K and has moreover a G-action (from the left).

<u>J. Klein</u>: If G is Lie, $S^{\mathfrak{g}} = \mathfrak{g} \cup \{\infty\}$ the one point compactification of the Lie algebra $\mathfrak{g}$ with the adjoint action.

<u>Theorem</u>

There is a homotopy equivalence $D\Sigma^\infty_+ G \wedge S^{\mathfrak{g}} \xrightarrow{\simeq} \Sigma^\infty_+ G$ in the category Sp_K, equivariant wrt G-action.

When $K = K(n)$, we see that $K(n)_*(D\Sigma^\infty_+ G \wedge \mathbb{S}^{\mathfrak{g}}) = K(n)_*(G$

$$\underbrace{K(n)_*(D\Sigma^\infty_+ G)}_{\displaystyle \operatorname{Hom}_{K(n)_*}(K(n)_* G, K(n)_*)} \otimes_{K(n)_*} K(n)_*(S^{\mathfrak{g}})$$

By looking at dimensions as $K(n)_*$-vector spaces we see that $\dim_{K(n)_*} K(n)_*(S^{\mathfrak{g}}) = 1$

Hovey-Strickland: This is equivalent to the existence of an inverse for smash product: $\exists\, Y$ s.t. $S^{\mathfrak{g}} \wedge Y = L_{K(n)}\mathbb{S}$.

By definition, $S^{\mathfrak{g}}$ belongs to the Picard group $\mathrm{Pic}(K(n)) \supseteq \hat{\mathbb{Z}}_p$ (+ a torsion part)

Example. $\mathrm{Pic}(H\mathbb{F}_p) = \mathbb{Z}$ (all sphere $\Sigma^n \mathbb{S}, n \in \mathbb{Z}$) are the $\wedge$-invertible spectra.

Def: The dimension $\dim(G)$ is the class $[S^{\mathfrak{g}}]$ in the Picard group $\mathrm{Pic}(K(n))$.

How do we compute the dimension? $n=1$, replace $K(1)$ by KU

$KU^*(\mathbb{S}^{2m})$. Use Adams operations to recover m:
$\psi^k : KU^0(\mathbb{S}^m) \xrightarrow{k^{2m}} KU^0(\mathbb{S}^m)$. How to deal with $n>1$?
There are operations $\psi^k : E_n^0(X) \longrightarrow E_n^0(X)$, $k \in \mathbb{Z}_p^\times$
~~stable~~ & unstable: $\psi^p : E_n^0(X) \longrightarrow E_n^0(X)$

$\psi^k : E_n^0(\mathbb{CP}^\infty) \longrightarrow E_n^0(\mathbb{CP}^\infty)$
generator $z \longmapsto [k]_F z = z +_F \cdots +_F z$.

Proposition
$\psi^k : E_n^0(\mathbb{S}^{2m}) \xrightarrow{\cdot k^m} E_n^0(\mathbb{S}^{2m})$

proof: $S^{2m} = S^2 \wedge \cdots \wedge S^2$. So we are done if we prove it for $m=1$ by Künneth isomorphism. But $S^2 \hookrightarrow \Sigma \mathbb{C}P^\infty$ is the bottom cell and we have seen that the effect on z in $E_n^0(\mathbb{C}P^\infty)$ is

$[F]_k z = kz + \alpha_2 z^2 + \ldots$. Projecting to S^2 gives $\psi^k(z) = k \cdot z$. $\square$

Lemma

If $\mathrm{Pic}(K(n)) \longrightarrow \mathbb{Z}_p^\wedge$, then ψ^k acts as
$X \longmapsto d$
k^d on $E_n^0(X)$.

Example: $p=2$, $n=1$, $X = \mathbb{R}P^\infty = \mathrm{Fib}(\mathbb{C}P^\infty \xrightarrow{\cdot 2} \mathbb{C}P^\infty)$

$KU^*(\mathbb{R}P^\infty) = KU^*[[z]]/_{[2]z}$

$= KU^*[z]/(2z+z^2)$

$\underset{\text{unit}}{k=3}$, $\psi^3(z) = \psi^3(L-1) = L^3 - 1 = z^3 + 3z^2 + 3z$ in $\mathbb{C}P^\infty$

in the quotient: $4z - 6z + 3z = z \implies \underline{\underline{d=0}}$.

Open: If $p>2$, $\mathbb{R}P^\infty$ is not a sphere, so the computation will be more difficult.

Question: Which spheres are $K(n)$-compact groups?
(Håkon Bergsaker, student of J. Rognes)

Theorem

If G is a $K(n)$-compact group, $G = L_{K(n)} S^{2m-1}$ and we have a mono $L_{K(n)} S^1 \overset{i}{\hookrightarrow} L_{K(n)} G = G$, then $m < N$ for some fixed $N = N(n,p)$.

proof: $p=2$, m odd. Consider the operations ψ^2, ψ^3, the unstable ones

$$E_n^0(BG) \xrightarrow{i^*} E_n^0(\mathbb{C}P^\infty)$$
$$\| $$
$$E_n^0[[y]] \longrightarrow E_n^0[[z]]$$
$$y \longmapsto z^m$$

$\psi^3(z) = r_1 z + r_2 z^2 + \dots$ and $r_1 = 3$

$\psi^2(z) = t_1 z + t_2 z^2 + \dots$ and $t_1 = 2$

$t_i \equiv \begin{cases} 1 \bmod (2, v_1, \dots, v_{n-1}) & \text{if } i = q = 2^i \\ 0 & \text{else} \end{cases}$

because the Formal Group law is the universal one of height n.

Restrict to $\psi^3(y) = \sum \rho_i y^i$, $\rho_1 = 3^m$

$\psi^2(y) = \sum \tau_i y^i$, $\tau_1 = 2^m$

$\tau_i \equiv \begin{cases} 1 & i = q \\ 0 & \text{else} \end{cases}$

We will derive a bound on m by looking at

$$\psi^3 \psi^2(y) = \psi^2\psi^3(y) = \sum \rho_i (\sum \tau_j y^j)^i$$
$$\| $$
$$\sum \tau_j (\sum \rho_i y^i)^j$$

Thus $\underset{3^m}{\rho_1} \tau_i + \rho_2 a_1 + \dots + \rho_{i-1} a_{i-1} + \rho_i 2^{im}$

$= \underset{2^m}{\tau_1} \rho_i + \tau_2 b_2 + \dots + \tau_{i-1} b_{i-1} + \tau_i 3^{im}$

and $a_i \in (\tau_1, \dots, \tau_{i-1})$

So $\tau_i(\underbrace{3^m}_{\substack{\text{unit} \\ \text{mod } 2}}(3^{(i-1)m} - 1)) = \rho_i 2^m(2^{(i-1)m} - 1) + \rho_2 a_2 + \dots + \rho_{i-1} a_{i-1}$
$- \tau_2 b_2 - \dots - \tau_{i-1} b_{i-1}$

Look at the order mod $\mathfrak{m} = (2, v_1, \dots, v_{n-1})$:

$$\operatorname{ord}(\tau_i) + \operatorname{ord}_2(3^{(i-1)m} - 1) \geq \min\{m, \operatorname{ord}(\tau_2), \dots, \operatorname{ord}(\tau_{i-1})\}$$

By induction, $\operatorname{ord}(\tau_i) \geq m - \sum_{j=1}^{i-1} \operatorname{ord}_2(3^{jm} - 1)$

When $i=q$: $m \leq \sum_{j=1}^{q-1} \operatorname{od}_2(3^{jm}-1) = 2^{n-1}\cdot 5 - n - 3$

<u>Remark</u>: We only compute the obstruction to be an A_q-space (not really a loopspace).

<u>Question 1</u>: Find maximal tori.

<u>Question 2</u>: Realize some spheres as $K(n)$-compact groups.

<u>Question 3</u>: Are there possibly infinite groups G acting on $L_{K(n)}BT$ such that $\Omega L_{K(n)}(BT_{hG})$ is a $K(n)$-compact group?