Nathalie: $\boxed{\text{String Topology of } BG\text{'s}}$, Chataur-Menichi + Westerland

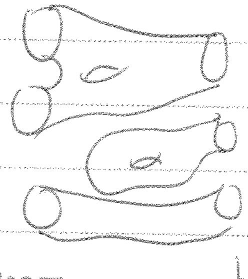
Fix a field $\mathbb{F}$ and let $X = BG$, G a finite group
or X a 1-connected space with $H^*(\Omega X;\mathbb{F})$ is finite dim (of dim d)
such as • BG, G connected compact Lie group
• p-compact groups for $\mathbb{F} = \mathbb{F}_p$

Theorem

For X as above, $H^*(LX;\mathbb{F})$ is an HCFT (positive boundary) where $LX = \mathrm{map}(S^1, X)$ (up to degree)

HCFT: Homological Conformal Field Theory

Let $\mathcal{C}$ be the topological category with $\mathrm{Obj}\,\mathcal{C} = \mathbb{N} =$ {closed 1-dim. manifolds}
$\mathrm{Mor}_{\mathcal{C}}(n,m) =$ moduli space of Riemann cobordisms from n to m circles such as in $\mathrm{Mor}_{\mathcal{C}}(3,3)$



i.e. the space of Riemann structures on such cob. $/\sim$

$$\simeq \coprod_{\substack{k, g_1,\dots,g_k \geq 0 \\ n_1+\dots+n_k = n \\ m_1+\dots+m_k = m}} BT_{g_1, n_1+m_1} \times \dots \times BT_{g_k, n_k+m_k}$$
where $T_{g,n} = \pi_0 \mathrm{Diff}(S_{g,n}, \mathrm{rel}\,\partial)$

$\mathcal{C}$ is a symmetric monoidal category under disjoint union

Def: A CFT is a monoidal functor

$$\Phi : \mathcal{C} \longrightarrow \mathcal{H}ilb$$
s.t. $\Phi(n+m) \cong \Phi(n) \otimes \Phi(m)$

Let $C_* \mathcal{C}$ be the linear category with same objects as $\mathcal{C}$ but $Mor_{C_*\mathcal{C}}(n,m) = C_*(Mor_{\mathcal{C}}(n,m); \mathbb{F})$

<u>Def</u>: A <u>TCFT</u> is a monoidal functor
$$\Phi : C_*\mathcal{C} \longrightarrow \text{Chain complexes}/\mathbb{F}$$
st. $\Phi(n+m) \cong \Phi(n) \otimes \Phi(m)$

Let $H_*\mathcal{C}$ be the linear category with same objects as $\mathcal{C}$ but $Mor_{H_*\mathcal{C}}(n,m) = H_*(Mor_{\mathcal{C}}(n,m); \mathbb{F})$.

<u>Def</u>. An <u>HCFT</u> is a monoidal functor
$$\Phi : H_*\mathcal{C} \longrightarrow Gr.Vect/\mathbb{F}$$
st. $\Phi(n+m) \cong \Phi(n) \otimes \Phi(m)$. (<u>Obs</u>: $\Phi(0) = \mathbb{F}$)

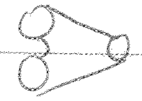
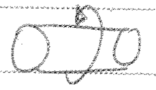
Let $\mathcal{C}^+ \subseteq \mathcal{C}$ be the subcategory with same objects and morphisms are such that every component of the cobordisms have non-empty incoming & outgoing boundary circle (ie $n_i, m_i > 0 \ \forall i$).

<u>Rephrased Theorem</u>
The map $n \longmapsto H_*(LX;\mathbb{F})^{\otimes n}$ can be extended to a monoidal functor from $H_*\mathcal{C}^+$ to graded $\mathbb{F}$-vector spaces.

In particular we have maps, $\forall g, n, m$
$$H_*(\Gamma_{g,n+m};\mathbb{F}) \otimes H_*(LX;\mathbb{F})^{\otimes n} \longrightarrow H_*(LX;\mathbb{F})^{\otimes n}$$
degree...

<u>Classical String Topology</u>

Chas-Sullivan: $H_{*+d}(LM)$ is a BV-algebra, ie there is a commutative product of degree zero and a degree one operator

Observe: The BV-structure is exactly the part of an HCFT coming from  giving the product and  of degree 1 giving Δ.

Theorem (Godin)
The Chas-Sullivan BV-structure extends to an HCFT on $H_*(LM)$

Back to $BG = X$

Consider the diagrams

① $(LX)^n \xleftarrow{\text{in}} \mathrm{Map}(S_{g,n+m}, X) \xrightarrow{\text{out}} (LX)^m$

these maps are $\mathrm{Diff}(S_{g,n+m}; \mathrm{rel}\,\partial)$-equivariant where Diff acts trivially on LX. Take thus Borel constructions

②
$$\begin{tikzcd} B\mathrm{Diff} \times (LX)^n & E\mathrm{Diff} \times_{\mathrm{Diff}} \mathrm{Map}(S_{g,n+m}, X) \arrow[l, "\pi"'] \arrow[r] \arrow[dr] & B\mathrm{Diff} \times (LX)^m \arrow[d, "\mathrm{proj}"] \\ & & (LX)^m \end{tikzcd}$$

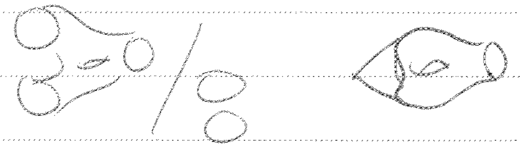
We want now transfer maps in homology.
If so, we get the HCFT by taking the composition.
We need to understand the fiber of π.

Proposition
$\mathrm{Map}(S, X) \xrightarrow{\text{in}} (LX)^n$ is a fibration with fiber $(\Omega X)^{-\chi(S)}$ if X is 1-connected

proof: It is a fibration because it is induced by a cofibration. Since LX is connected (X is 1-connected), we look at the fiber over the trivial loops $\coprod S^1 \xrightarrow{c} X$. It is

$$\mathrm{Map}_*(S/\textstyle\coprod S^1, X), \text{ but } S_{g,n+m}/\coprod_n S^1 \simeq \bigvee_{\underbrace{2g+m-1+n-1}_{-\chi(S)}} S^1$$

Ex:



$\square$

Corollary: The fiber of π is $(\Omega X)^{-\chi(S)}$. $\square$

Since we assume $H_*(\Omega X;\mathbb{F})$ is finite, the Dwyer transfer exists.

Note: If G is finite, since $S_{g,n+m} \simeq \bigvee S^1$, we have $\mathrm{map}(\bigvee S^1, BG)$ easy to describe ...

Integration along fibers

If $H_d(F;\mathbb{F}) \cong \mathbb{F}$ in top dim., $\pi_1 B$ acts trivially on $H_d(F;\mathbb{F})$, the SS for $F \to E \to B$ gives

$$H_p(B) \xrightarrow{\cong} E^2_{p,d} = H_p(B; H_d(F;\mathbb{F}))$$

$$\downarrow$$

$$E^\infty_{p,d}$$

$$\cap$$

$$H_{p+d}(E)$$

In this case we get $d = \underbrace{\chi(S)}_{\text{depends on } g,n,m} \cdot \dim \Omega X$

• the orientability condition holds as well since we have a pull-back

$$\begin{tikzcd} \mathrm{Map}(S, X) \arrow[r, "in"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (LX)^n \arrow[d, "ev"] & \text{orientable} \\ \mathrm{Map}(\bigvee, X) \arrow[r] & X^n & \text{— orientable} \arrow[u, Rightarrow] \end{tikzcd}$$