(1) <u>Natalia Castellana: p-Noetherian groups, part B</u>

joint work with Jerôme & Chiqui

$\mathcal{U}$ = category of unstable modules over $\mathcal{A}_p$

There is a Krull filtration: $\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \ldots \subseteq \mathcal{U}_n \subseteq \ldots \subseteq \mathcal{U}$

[Schwartz]: $M \in \mathcal{U}_m \Leftrightarrow \bar{T}^{m+1} M = 0$

$\bar{T}$ is left adjoint to $- \otimes \tilde{H}^*(B\mathbb{Z}/p)$, $\bar{T}: \mathcal{U} \to \mathcal{U}$

It satisfies $TM = M \oplus \bar{T}M$

<u>Note</u>: $QH^*(K(\mathbb{Z}/p, n+1)) \in \mathcal{U}_n$

<u>Note</u>: $M \in \mathcal{U}_0 \Leftrightarrow M$ locally finite $\mathcal{A}_p$-module, ie $x \in M \Rightarrow \mathcal{A}_p x$ finite

<u>Note</u>: $H^*(Z)$ fg $\mathcal{A}_p$-algebra

$QH^*(Z)$ fg $\mathcal{A}_p$-module $\Rightarrow QH^*Z \in \mathcal{U}_n$ for some n.

<u>Note</u>: $TQH^*(Z) \cong Q(TH^*(Z))$ (exactness of T, write QH^*Z as quotient of $\bigoplus_i F(n_i)$)

fg $\mathcal{A}_p$-alg

$\Rightarrow TH^*Z$ fg $\mathcal{A}_p$-alg

$\Rightarrow TH^*Z$ finite type with

From previous work ~~of~~ Jerôme & Chiqui:

<u>Thm</u>: Z connected H-space, $T_V H^*Z$ finite type (automatic if H^*Z fg as an $\mathcal{A}_p$-algebra).

$QH^*Z \in \mathcal{U}_n \Leftrightarrow \Omega^{n+1} Z$ is $B\mathbb{Z}/p$-local, ie $\mathrm{map}_*(B\mathbb{Z}/p, \Omega^{n+1}Z) \simeq *$.

<u>Note</u>: This is due to Dwyer-Wilkerson for $n=0$.

<u>Key property in proof of $\Leftarrow$</u>: Use geometrical interpretation of

T: $BV \times \mathrm{map}(BV, Z) \xrightarrow{ev} Z$

$H^*(Z) \longrightarrow H^*(BV) \otimes H^*(\mathrm{map}(BV, Z))$,

$T_V H^*(Z) \longrightarrow H^*(\mathrm{map}(BV, Z))$ isomorphism under certain conditions.

(2) $\mathrm{map}_*(BV, Z) \longrightarrow \mathrm{map}(BV, Z) \underset{s}{\overset{ev}{\rightleftarrows}} Z$ is an H-fibration

$\mathrm{map}(BV, Z) \simeq \mathrm{map}_*(BV, Z) \times Z$

H-spaces are p-good, so $T_V H^*(Z) \cong H^*(\mathrm{map}(BV, Z))^\wedge$

$\cong H^*(\mathrm{map}_*(BV, Z)) \otimes H^*(Z)$

Hence

$T_V(Q(H^* Z)) \cong Q(T_V(H^* Z))$

$\cong Q(H^*(\mathrm{map}_*(BV, Z))) \oplus Q(H^*(Z))$

So $Q(H^*(\mathrm{map}_*(BV, Z))) \simeq \overline{T}(Q(H^* Z))$. (geometrical interpretation of $\overline{T}$).

Z H-space

Want: More general

(1) Geometrical interpretation of $\overline{T}$ under certain conditions (but not H-space condition)

(2) Information on $\mathrm{map}_*(B\mathbb{Z}/p, Z)$

Observation: $Q\,\overline{T}(H^* Z) \cong Q\,\overline{T}_C(H^*(Z))$

We will look at $\mathrm{map}_*(B\mathbb{Z}/p, Z)_{\mathrm{konst}} \longrightarrow \mathrm{map}(B\mathbb{Z}/p, Z)_{\mathrm{Const}} \xrightarrow{ev} Z$

$\boxed{\text{From now on } Z \text{ is 1-connected}}$

Question: Can we find conditions under which the above fibration splits?

(3) If (X, BX, e) is a p-Noetherian group with X connected, ie BX 1-connected.

<u>Prop</u>: There exists a fibration

$K(P,2) \to BX \to BY^\wedge_p$ where P = finite sum of copies of $\mathbb{Z}/p^\infty$ and $\mathbb{Z}/p^{n_i}$

and $H^*(BY)$ is noetherian.

<u>Proof</u>: Look at

$$BX \longrightarrow P_{\Sigma B\mathbb{Z}/p}(BX)$$

Ω

$$F \longrightarrow \underset{\substack{\text{noetherian}\\\text{cohomology}}}{X} \longrightarrow \Omega P_{\Sigma B\mathbb{Z}/p}(BX) = P_{B\mathbb{Z}/p}X$$

Bousfield + earlier work of Natalia, Chogui, Jerome:
(theory of H-spaces)
$F \simeq K(P,1)$ $\xrightarrow{\text{finiteness conditions}}$ P = finite direct sum as above.

By results of Chogui (thesis):

$H^*(P_{B\mathbb{Z}/p}X)$ finite
(X loop space).

Thus set $Y = P_{B\mathbb{Z}/p}X$, $Y^\wedge_p$ p-compact group $B(Y)^\wedge_p \simeq (P_{\Sigma B\mathbb{Z}/p}BX)^\wedge_p$

$\Rightarrow H^*BY$ noetherian

QUESTION: X finite Postnikov piece. Do we have H^*X fg as algebra / $\mathcal{A}_p$??

(4) Since BY is $\Sigma B\mathbb{Z}/p$-local,

$$map_*(\Sigma B\mathbb{Z}/p, K(P,2)) \simeq map_*(\Sigma B\mathbb{Z}/p, BX)$$

$W = Hom(\mathbb{Z}/p, P) \simeq map_*(B\mathbb{Z}/p, K(P,1)) \simeq map_*(B\mathbb{Z}/p, \Omega BX)$ is an elementary abelian p-group.

$$map_*(B\mathbb{Z}/p, BX)_{const} \cong K(W,1).$$

Point 1

Study fibrations $F \to E \to B$, where
$F \simeq F[n]$, ie $\pi_i F = 0$ $i > n$.

Thm: $F \to E \to B$ connected space, $F \simeq F[n]$

(a) If B is $(n+1)$-connected, then $E \simeq F \times B$

(b) If B is n-connected and there exists a section then $E \simeq F \times B$.

Proof sketch: classified by $Baut(F)$ and $Baut_*(F)$ if there is a section, ie

(a) $B \to Baut(F)$

(b) $B \to Baut_*(F)$

Prop: $Baut(F) \simeq Baut(F)[n+1]$
$Baut_*(F) \simeq Baut_*(F)[n]$

Examples $(n+1)$ respectively n is optimal

(a) $K(\mathbb{Z}, n) \to S^{n+1}\langle n+1 \rangle \to S^{n+1}$

(b) $F = K(\mathbb{Z}/p, 1) \times K(\mathbb{Z}/p, n)$ ...

$\pi_n(Baut F)) \cong \mathbb{Z}/p$

(5) <u>Corollary</u>: $n \geq 1$, A space, X n-connected such that $\Omega^{n+1}X$ is A-local. Then $map(A,X)_{const} \simeq map_*(A,X)_{const} \times X$.

<u>Prop</u>: Y p-complete, n-connected with $\Omega^{n+1}Y$ $B\mathbb{Z}/p$-local. Assume $T_{V,const}(H^*(Y))$ finite type, $map(BV,Y)_{const}$ p-good. Then $\overline{T}_V(Q(H^*(Y))) \cong Q(H^*(map_*(BV,Y)_{const}))$

(X,BX,e) p-noetherian group, X connected

- $\Omega^2 BX$ $B\mathbb{Z}/p$-local, $map_*(B\mathbb{Z}/p, BX)_{const} \simeq K(W,1)$ p-good

- Fibration: $K(W,1) \longrightarrow map(B\mathbb{Z}/p, BX)_{const} \overset{s}{\longrightarrow} BX$

1-connected

$\rightsquigarrow map(B\mathbb{Z}/p, BX)_{const} \simeq K(W,1) \times BX$ is p-good

- $H^*(BX)$ fg algebra$/A_p \rightarrow T_{V,const}(H^*(BX))$ finite type

Hence the conditions in the above proposition hold and

$$\overline{T}_V(Q(H^*(BX))) \cong Q(H^*(map_*(B\mathbb{Z}/p, BX)_{const})) = \underbrace{Q(H^*(BW))}_{\text{finite}}$$

So $\overline{T}_V(\overline{T}_V(Q(H^*(BX)))) = 0$ so $Q(H^*(BX)) \in \mathcal{U}_1$.

(6) Remark (Kac-Moody groups)

$Q\left(H^*BK_A\langle 4\rangle\right) \in \mathcal{U}_1$

A ~~Indecomposable~~ Generalized Cartan Matrix

Remark In general we have

THM X $(n-1)$-connected, $H^*(X)$ noetherian $\Longrightarrow$
$n \geq 2$

$$\left\{\begin{array}{l} H^*(X\langle n\rangle) \text{ fg } \mathcal{A}_p\text{-algebra} \\ QH^*(X\langle n\rangle) \in \mathcal{U}_{n-2} \end{array}\right\}$$