(1) <u>Def</u> An $n\times n$-matrix A is called a <u>Generalized Cartan Matrix</u> (GCM) if

- $a_{ij}\in\mathbb{Z}$; $a_{ii}=2$; $a_{ij}\leq 0$ for $i\neq j$; $a_{ij}=0 \Leftrightarrow a_{ji}=0$

<u>Def</u> A is <u>symmetrizable</u> if there exist invertible diagonal matrix D (over $\mathbb{Q}$) such that AD is symmetric. $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ ~~indecomposable~~ symmetric (indecomp, FIN, type A_2)

<u>Ex</u>: $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ is symmetric (indecomposable) (AFF, type $A_2^{(1)}$)

$\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ is not symmetrizable (indecomposable) (IND)

<u>Definition</u>: A $n\times n$ GCM A is called <u>decomposable</u> if $\exists\, \sigma\in\Sigma_n : \left(A_{\sigma(i),\sigma(j)}\right)_{ij} = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$ where A_1 and A_2 have ≥ 1 rows.

Otherwise A is called <u>indecomposable</u>.

~~Theorem (Vinberg 1971) Let A be an indecomposable GCM. Then~~

<u>Notation</u>: For $v\in\mathbb{R}^n$ we write $v>0 \Leftrightarrow v_i>0$ for $i=1,\dots,n$
≥ 0 — ≥ 0

<u>Theorem (Vinberg 1971)</u> Let A be an indecomposable GCM. Then we have precisely one of the following three cases:

(FIN) $\exists v>0 : Av>0$.

(AFF) $\exists v>0 : Av=0$.

(IND) $\exists v>0 : Av<0$.

Moreover in these cases we have the extra information

(FIN) $\det A \neq 0$; $Av \geq 0 \Rightarrow v>0$;
(all principal minors >0) $\Leftrightarrow$ FIN

(AFF) $\det A=0$; corank $A=1$; $Av \geq 0 \Rightarrow Av=0$;
(all proper principal minors >0 + $\det A=0$) $\Leftrightarrow$ AFF

(IND) $\det A \neq 0$; $Av \geq 0, v \geq 0 \Rightarrow v=0$

<u>Theorem</u>: Let A be an indecomposable GCM of type (FIN). Then A is one of the following:

A_l, $l \geq 1$; B_l $l \geq 2$; C_l $l \geq 3$; D_l $l \geq 4$; E_6, E_7, E_8; F_4; G_2.

<u>Construction of Extended Cartan Matrix</u>
Let $\mathring{A}$ be an indecomposable GCM of type (FIN). Let Δ be the associated root system (see later for DEF!) and $\tilde{\alpha}$ the highest root. Define $(A_{ij})_{0 \leq i,j \leq n}$ by • $a_{ij} = \mathring{A}_{ij}$ $1 \leq i,j \leq n$

$\tilde{\alpha}^\vee$ corresponding coroot

• $a_{00} = 2$

• $a_{0i} = -\alpha_i(\tilde{\alpha}^\vee)$ • $a_{i0} = -\tilde{\alpha}(\alpha_i^\vee)$ $1 \leq i \leq n$.

(3) Then A is a GCM of type (AFF), ~~that~~ denoted $\mathring{A}^{(1)}$

(more precisely $\mathring{A}$ of type say E_6, A is ~~said~~ to have type $E_6^{(1)}$

<u>Theorem</u>: Let A be an indecomposable GCM of type AFF. Then A is either

- $\mathring{A}^{(1)}$ for some ~~A~~ indecomposable GCM $\mathring{A}$ of type (FIN) (untwisted affine case)
- $\underset{\text{size } 2}{A_2^{(2)}}$, $\underset{\text{size } l+1}{A_{2l}^{(2)}}$ $l \geq 2$; $\underset{\text{size } l+1}{A_{2l-1}^{(2)}}$ $l \geq 3$; $\underset{\text{size } l+1}{D_{l+1}^{(2)}}$ $l \geq 2$
  $\underset{\text{size } 5}{E_6^{(2)}}$, $\underset{\text{size } 3}{D_4^{(3)}}$ (twisted affine cases)

(4) Associated Lie algebras

Def Let A be a $\overset{l\times l}{\vee}$ GCM. A realization of A is a triple $(\mathfrak{h}, \pi, \pi^\vee)$ where

- $\mathfrak{h}$ is a vector space$/\mathbb{C}$ of dim $2l - \operatorname{rank} A$
- $\pi = \{\alpha_i\}_{i=1}^{l} \subseteq \mathfrak{h}^*$ independent set
- $\pi^\vee = \{\alpha_i^\vee\}_{i=1}^{l} \subseteq \mathfrak{h}$ independent set
- $\alpha_j(\alpha_i^\vee) = a_{ij}$

Prop: Realizations exist and are unique up to isomorphism

$(\text{ie } \exists \theta : \mathfrak{h}_1 \xrightarrow{\cong} \mathfrak{h}_2,\ \theta(\alpha^\vee_{i,1}) = \alpha_{i,2},\ \theta^*(\alpha_{i,2}) = \alpha_{i,1})$

Def: Let A be an $l \times l$ GCM, with realization $(\mathfrak{h}, \pi, \pi^\vee)$. Then $\tilde{\mathfrak{g}}(A)$ is the Lie algebra generated by $\mathfrak{h}$ and symbols e_i, f_i, $1 \le i \le l$ modulo the relations.

$$(R)\quad [\mathfrak{h}, \mathfrak{h}] = 0$$

$$\left.\begin{aligned} [h, e_i] &= \alpha_i(h) e_i \\ [h, f_i] &= -\alpha_i(h) f_i \end{aligned}\right\} h \in \mathfrak{h},\ 1 \le i \le l$$

$$[e_i, f_j] = \begin{cases} \alpha_i^\vee & i = j \\ 0 & i \ne j \end{cases}$$

Consider also the Serre relations

$$(S)\quad \left.\begin{array}{l} (\operatorname{ad} e_i)^{1-a_{ij}}(e_j)=0 \\ (\operatorname{ad} f_i)^{1-a_{ij}}(f_j)=0 \end{array}\right\} \text{ if } i\neq j$$

<u>Prop</u> There is a largest ideal $\mathfrak{r}$ of $\tilde{\mathfrak{g}}(A)$ with $\mathfrak{r}\cap\mathfrak{h}=0$. The Serre relations are contained in $\mathfrak{r}$.

So we have 3 Lie algebras

$$\tilde{\mathfrak{g}}(A) \twoheadrightarrow \tilde{\mathfrak{g}}(A)/\text{Serre relations} \twoheadrightarrow \tilde{\mathfrak{g}}(A)/\mathfrak{r}$$

<u>Kac</u>		$\mathfrak{g}(A)$
<u>Kumar</u>	$\mathfrak{g}(A)$	$\overline{\mathfrak{g}}(A)$

<u>Thm (Gabber-Kac '81)</u> If A is symmetrizable, then $\mathfrak{r}=\langle\text{Serre rels}\rangle$ so $\mathfrak{g}(A)=\overline{\mathfrak{g}}(A)$.

<u>Remark</u>: Open in general!

<u>Structure of $\tilde{\mathfrak{g}}(A)$</u>

<u>Def</u> $Q=\bigoplus_{i=1}^{n}\mathbb{Z}\alpha_i$ root lattice

$Q_+=\bigoplus_{i=1}^{n}\mathbb{Z}_{\geq 0}\alpha_i$

$\tilde{\mathfrak{n}}_+ =$ subalg gen by $e_1,\dots,e_l$

$\tilde{\mathfrak{n}}_- =$ - - - - - - $f_1,\dots,f_l$

~~Thm~~
~~• $\tilde{\mathfrak{g}}=\tilde{\mathfrak{n}}_-\oplus\mathfrak{h}\oplus\tilde{\mathfrak{n}}_+$~~
~~• $\tilde{\mathfrak{n}}_+$ free on $e_1,\dots,e_l$~~
~~• $\tilde{\mathfrak{n}}_-$ - - $f_1,\dots,f_l$~~

(6) Thm

- $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$
- $\tilde{\mathfrak{n}}_\pm$ free on $\{e_1, \dots, e_l\} / \{f_1, \dots, f_l\}$
- $\tilde{\mathfrak{g}} = \left(\bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \tilde{\mathfrak{g}}_{-\alpha} \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \tilde{\mathfrak{g}}_{\alpha} \right)$

where we for $\alpha \in \mathfrak{h}^*$ set $\tilde{\mathfrak{g}}_\alpha = \{ x \in \tilde{\mathfrak{g}} \mid [h, x] = \alpha(h) x \text{ for all } h \in \mathfrak{h} \}$.

We have $\dim \tilde{\mathfrak{g}}_\alpha < \infty$ for all $\alpha \in \mathfrak{h}^*$

Similarly for $\mathfrak{g}$.

Def: $\Delta = \{ \alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0, \alpha \neq 0 \}$

Prop: Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. We have

- $\mathfrak{g} = \mathfrak{g}' + \mathfrak{h}$, $\mathfrak{h} \cap \mathfrak{g}' = \bigoplus_{i=1}^{l} \mathbb{C} \alpha_i^\vee$

Weyl group

For $1 \leq i \leq l$ define $s_i \in \mathrm{Aut}(\mathfrak{h})$,

$s_i(h) = h - \alpha_i(h) \alpha_i^\vee$

$W = \langle s_i \mid i = 1, \dots, l \rangle \leq \mathrm{Aut}(\mathfrak{h})$ Weyl group of $\mathfrak{g}(A)$

W is a Coxeter group, ie $W \cong \langle S_i \mid i = 1, \dots, l \mid (S_i S_j)^{m_{ij}} \rangle$ where

$a_{ij} \cdot a_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

Crystallographic Coxeter gp.

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__Remark__ A indecomposable GCM. Then $|W| < \infty \Leftrightarrow A$ (FIN)
$\Leftrightarrow |\Delta| < \infty$

__Example, affine case__

Let $\mathring{A}$ ($\ell \times \ell$) be an indecomposable GCM of type (FIN).

Define the Extended Cartan Matrix A ($\ell+1 \times \ell+1$) as above.
Consider the loop algebra

$$\mathcal{L}(\mathring{A}) = \mathfrak{g}(\mathring{A}) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$$

$$\hat{\mathcal{L}}(\mathring{A}) = \mathcal{L}(\mathring{A}) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with an explicit Lie bracket. Then we have

__Thm__ (Garland, Kac...) $\mathfrak{g}(A) \cong \hat{\mathcal{L}}(\mathring{A})$, $\hat{\mathcal{L}}'(A) = \mathcal{L}(\mathring{A}) \oplus \mathbb{C}c$, there is a universal central extension $0 \to \mathbb{C}c \to \hat{\mathcal{L}}'(A) \to \mathcal{L}(\mathring{A}) \to 0$.

Roots

Δ^{re} (real roots) $= W \cdot \{\alpha_1, \dots, \alpha_\ell\}$

Δ^{im} (imaginary roots) $= \Delta \setminus \Delta^{re}$

Prop: For $\alpha \in \Delta^{re}$ we have $\dim \mathfrak{g}_\alpha = 1$, for $z \in \mathbb{C}$

$z\alpha \in \Delta \Rightarrow z = \pm 1$

- For $\alpha \in \Delta^{im}$: $n\alpha \in \Delta$ for $n \in \mathbb{Z}, n \neq 0$

Rem If A is an indecomposable then

$\Delta^{im} = \emptyset \Leftrightarrow A$ (FIN)

$\underbrace{\Delta^{im} = \mathbb{Z}\delta}_{1\text{-dim}} \Leftrightarrow A$ (AFF)

"larger" $\Leftrightarrow A$ (IND)

(9)

Integrable representations

~~Let $\mathfrak{g}$ be a complex Lie algebra, $\pi : \mathfrak{g} \to \mathrm{Aut}(V)$~~

V a complex vector space

<u>Def</u>: $f : V \to V$ is <u>locally finite</u> $\iff$

$\dim_{\mathbb{C}} \operatorname{span}\{v, f(v), f^2(v), \dots\} < \infty \quad \forall v \in V$

<u>f is locally nilpotent</u> $\iff \forall v \in V \; \exists n \in \mathbb{N} : f^n(v) = 0.$

$\mathfrak{g}$ complex Lie algebra, $\pi : \mathfrak{g} \to \mathrm{Aut}(V)$ representation

$F_{\mathfrak{g}} = \{x \in \mathfrak{g} \mid \operatorname{ad} x : \mathfrak{g} \to \mathfrak{g} \text{ locally finite}\}$

$F_{\mathfrak{g},\pi} = \{x \in F_{\mathfrak{g}} \mid \pi(x) : V \to V \text{ locally finite}\}$

<u>Prop</u> subalg of $\mathfrak{g}$ gen by $F_{\mathfrak{g},\pi} = \operatorname{span}_{\mathbb{C}} F_{\mathfrak{g},\pi}$

<u>Def</u> $\mathfrak{g}$ <u>integrable</u> $\iff \mathfrak{g} = \operatorname{span}_{\mathbb{C}} F_{\mathfrak{g}}$

(π, V) <u>integrable</u> $\iff F_{\mathfrak{g},\pi} = F_{\mathfrak{g}}$

<u>Prop</u> $\mathfrak{g}(A)$ is integrable

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<u>Construction (The group associated to $\mathfrak{g}$)</u>

Let $\mathfrak{g}$ be integrable, $G^* :=$ free group on $F_{\mathfrak{g}}$

Let $\pi : \mathfrak{g} \to \mathrm{Aut}(V)$ be integrable. Then G^* acts on V by

$$\rho_\pi : G^* \to \mathrm{Aut}(V), \quad \rho_\pi(x) = \exp(\pi(x)) = \sum_{n\geq 0} \frac{1}{n!} \pi(x)^n \in \mathrm{Aut}(V), \quad x \in F_{\mathfrak{g}}$$

$$N^* = \bigcap_{\substack{\pi : \mathfrak{g} \to \mathrm{Aut}(V) \\ \text{integrable}}} \mathrm{Ker}\, \rho_\pi$$

$G = G^*/N^*$, the group associated to $\mathfrak{g}$.

<u>Example</u> $\mathfrak{g}$ simple, $\dim \mathfrak{g} < \infty$

Then G = associated simply connected Lie group.

<u>Thm (Kac-Peterson)</u>: $\mathfrak{g} = \mathfrak{g}(A)$. Then $\pi : \mathfrak{g} \to \mathrm{Aut}(V)$ integrable

$\Longleftrightarrow$ $\pi(e_i), \pi(f_i) : V \to V$ are locally nilpotent.

<u>Def</u> $G(A)$ = the group associated to $\mathfrak{g}'(A)$.

~~Affine case There is a universal central extension~~

<u>Affine case</u> $G^\circ_{\mathbb{L}}$ = grp associated to $\mathring{\mathfrak{g}}(\mathring{A}) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \simeq$

$\mathring{A}$ indecomposable (FIN) $\quad \{\text{polynomial maps } \mathbb{C}^* \to \mathring{G}\} \simeq \Lambda G^0$

(11) There is a universal central extension

$$1 \to \mathbb{C}^* \to G(A) \longrightarrow G^0_{\mathbb{L}} \to 1$$

Unitary (compact) form: There is a suitable involution ω of $\mathfrak{g}(A)$, G

$$K(A) = G(A)^{\omega}$$

$$K^0_{\mathbb{L}} = \{\text{polynomial loops } f: S^1 \to G^0\} \simeq \Lambda G^0$$

$$\tilde{K}^0_{\mathbb{L}} = \{\text{polynomial} \text{ - - - } , f(1)=e\} \simeq \Omega G^0$$

$$1 \to S^1 \to K(A) \xrightarrow{\tau} K_{\mathbb{L}} \to 1$$

$$K^0_{\mathbb{L}} = \tilde{K}^0_{\mathbb{L}} \rtimes G^0 \qquad \tau^{-1}(\tilde{K}^0_{\mathbb{L}}) \simeq \Omega G^0 \langle 2 \rangle$$

$$\boxed{K(A) = \Omega(G^0)\langle 2 \rangle \rtimes G^0}$$

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Algebraic construction