Then $G^\circ_{\mathbb{I}} \simeq \Lambda G^\circ$, free loop space.

<u>Theorem</u>
There is a universal central extension for $A = \mathring{A}^{(1)}$

$$1 \longrightarrow \mathbb{C}^* \longrightarrow G(A) \longrightarrow G^\circ_{\mathbb{I}} \longrightarrow 1$$

where $G(A)$ is the group associated to $\mathfrak{g}'(A)$.

Loop groups (related to affine Kac-Moody groups)

Let $G = \mathring{G}$ be a compact Lie group, simple and simply connected.
~~Define~~ the <u>loop group</u> $LG = \mathrm{map}_{sm}(S^1, G)$, space of smooth maps. The inclusion
$LG \hookrightarrow \Lambda G$ is a homotopy eq. (dense).
One could also take piecewise smooth, or analytic (by Peter-Weyl, embed $G \subset U(n)$ and look at analytic maps when composed into $U(n)$, i.e. the matrix ~~has~~ coefficients of the form $\sum_{k=-\infty}^{\infty} \gamma_k z^k$), or even polynomial maps.

<u>Example</u>: $G = U(n)$, $G_{\mathbb{C}} = GL_n(\mathbb{C})$. Then

$$L_{poly} G_{\mathbb{C}} = \mathrm{map}_{poly}(\mathbb{C}^\times, G_{\mathbb{C}}) = GL_n(\mathbb{C}[z, z^{-1}])$$

<u>Proposition</u>
Viewing LG as an ∞-dimensional manifold, and thus Lie group, $\mathrm{Lie}(LG) = L\mathfrak{g} = \mathrm{map}(S^1, \mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra associated to G and the bracket is defined pointwise.

So $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ where the bracket

$$[x \otimes z^m, y \otimes z^n] = [x, y] \otimes z^{m+n}.$$

We want an assignment $(\mathrm{Fin}) \longrightarrow (\mathrm{AFF})$

$$A \longmapsto \tilde{A} .$$

Central extensions

Let G be simple and 1-connected

Proposition

There exists a universal central extension

$$1 \to \mathbb{T} \to \widetilde{LG} \to LG \to 1$$

where topologically $\widetilde{LG} = (LG)\langle 2 \rangle$ and $\mathbb{T} = S^1$.

Note: Consider the split fibration $\Omega G \to LG \leftrightarrows G$.

In particular $LG \simeq \Omega G \times G$ as space, not as group

Since $\pi_1 G = 0 \Rightarrow \pi_2 G = 0$

$$\pi_3 G \cong \mathbb{Z}$$

The generator of $H_4(BG) \cong \mathbb{Z}$ is the unique symmetric G-invariant bilinear form on $\mathfrak{g}$. (the Killing form)

Thus LG is 1-connected and $H_2(LG, \mathbb{Z}) \cong \mathbb{Z}$.

This explains why the above extension exists.

At the level of Lie algebras, the 2-cocycle is given by the Killing form on $L\mathfrak{g}$

$$\omega_K(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\langle \xi(\theta), \eta'(\theta) \rangle_{\mathfrak{g}}}_{\text{Killing form}} \, d\theta$$

Affine Kac-Moody group

$\widetilde{LG}$ is the Kac-Moody group associated to the commutator of the affine Kac-Moody algebra constructed from G.

$\widetilde{LG} \rtimes S^1$ is the Kac-Moody group associated to $\mathcal{G}(\tilde{A})$, A a GCM of type (FIN).
The action is given by rotation of loops.

Observation: $L\mathcal{G}_{\mathbb{C}} = \bigoplus_{\mathbb{Z}} \mathcal{G} z^k$ is a decomposition as S^1-modules (rotation does not change the degree)

If T is a maximal torus in G, then the actions of T and S^1 commute. From Lie theory we know that

$$\mathcal{G}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathcal{G}_\alpha \quad (T\text{-action splitting}).$$

Thus $$L\mathcal{G}_{\mathbb{C}} \cong \bigoplus_{\mathbb{Z}} \mathfrak{h} z^k \oplus \bigoplus_{\alpha \in \Delta} \mathcal{G}_\alpha z^k$$

(splitting wrt $T \times S^1$-action).

Now $Lie(\widetilde{LG} \rtimes S^1) = \underset{\text{central } S^1}{\mathbb{C}c} \oplus \underset{\text{rotating } S^1}{\mathbb{C}d} \oplus L\mathcal{G}$ with sure bracket

The maximal torus in $\widetilde{LG} \rtimes S^1$ is $\mathbb{T} \times T \times S^1$

rank $\widetilde{LG} \rtimes S^1 = \text{rank } G + 2 = \text{rk} A + 2 = \text{size } \tilde{A} + 1$.

Note: The "+1" is forced by the fact that $\tilde{A}$ is degenerate and one cannot get a pairing between roots and coroots in dim = size $\tilde{A}$

$\mathrm{Lie}(\tilde{L}G \rtimes S^1)$ decomposes, as $\mathbb{T}\times T\times S^1$-modules,

$$\underbrace{\mathbb{C}c \oplus \mathbb{C}d \oplus \mathbb{C}\mathfrak{h}}_{\text{new } \mathfrak{h}} \oplus \overbrace{\bigoplus_{k\neq 0} \mathfrak{h} z^k}^{\text{large mult.}} \oplus \bigoplus_{\substack{\mathbb{Z} \\ \alpha\in\Delta}} \mathfrak{g}_\alpha z^k$$

What are the roots of this Lie algebra?

$$\tilde{\Delta} = \{\underbrace{k\alpha_0 \mid k\in\mathbb{Z}\smallsetminus 0}_{\text{of multiplicity } \ell}\} \cup \{\underbrace{k\alpha_0+\alpha \mid k\in\mathbb{Z}, \alpha\in\Delta}_{\text{of mult. } 1}\}$$

α_0 correspond to $\mathbb{T}\times T\times S^1 \longrightarrow S^1$, trivial on c, d

Algebraically, if $\alpha_1, \dots, \alpha_\ell$ are the simple roots, there exist a unique highest root (largest combination of simple root), and a lowest root $= -$ highest root $= \alpha_0$. The affine root system is the root system generated by $\{\alpha_0, \alpha_1, \dots, \alpha_\ell\}$

Example: $G = SU(\ell+1)$, roots are $\varepsilon_i - \varepsilon_j$ in $\mathbb{R}^{\ell+1}$

simple roots $\varepsilon_1-\varepsilon_2, \dots, \varepsilon_\ell - \varepsilon_{\ell+1}$

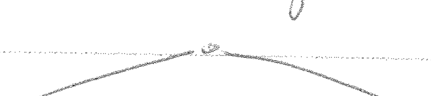
minimal root is $\varepsilon_{\ell+1}-\varepsilon_1 = -(\varepsilon_1-\varepsilon_2) - \dots - (\varepsilon_\ell - \varepsilon_{\ell+1})$

The Cartan matrix

$$\left(\begin{array}{c|cccc} 2 & -1 & 0 & \cdots 0 & -1 \\ \hline -1 & 2 & -1 & & \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & \ddots & -1 \\ -1 & 0 & & -1 & 2 \end{array}\right)$$

$$a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}.$$

The associated Dynkin diagram is



Def: ① The Weyl group is given by reflections in the roots
$$s_{\alpha_i}(x) = x - \alpha_i(x)\alpha_i^\vee$$

② $W_{aff} = N(\mathbb{T}\times T\times S^1)/_{\mathbb{T}\times T\times S^1}$ is the Weyl group.

Proposition

$W_{aff} = \check{T} \rtimes W$ (which explains the name "affine"

where $\check{T} = \pi_1(T)$