w.

p-Noetherian groups

(Jerome Scherer).
+Natalia Castellana ((CCS

Def: A p-Noetherian group is a triple (X, BX, e) where $H^*(X;\mathbb{F}_p)$ is noetherian. BX is p-complete. $e: \Omega BX \xrightarrow{\simeq} X$

Ex: ① $(K(\mathbb{Z}_p,2), K(\mathbb{Z}_p,3), e)$ is a p-noetherian

(note $H^*(K(\mathbb{Z},3);\mathbb{F}_2) = \mathbb{F}_2[\iota_3, Sq^2\iota_3, Sq^{4,2}\iota_3, \dots]$ is a f. g. and

⓪ p-compact groups ——— || ———

$QH^*(K(\mathbb{Z},3))$ $\quad \iota_3 \xrightarrow{Sq^2} \bullet \xrightarrow{Sq^4} \bullet$

is almost $\Sigma F(1)$

(more precisely $QH^*(K(\mathbb{Z},3)) \subseteq \Sigma F(1)$

and only difference is that the class in degree 2 is missing "$\Sigma Sq^1\iota_1$".)

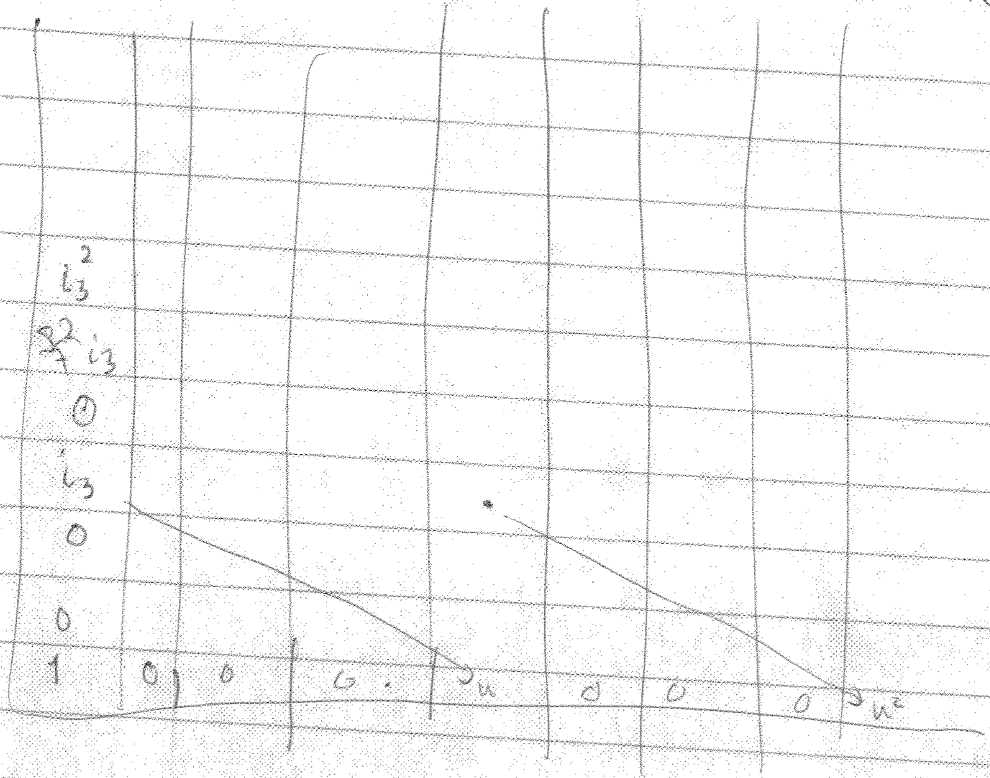
Example ① is the only example with ΩX finite by Kubshucks thm.

② $(S^3\langle 3\rangle^\wedge_p, (BS^3)\langle 4\rangle^\wedge_p, e)$

$p=2$: $H^*(S^3\langle 3\rangle) \Rightarrow \mathbb{F}_2[u_4] \otimes \Lambda(v_5)$ $\quad u_4 \xrightarrow{\beta = Sq^1} v_5$

$H^*(B(S^3\langle 3\rangle))$

$K(\mathbb{Z},3) \to B(S^3\langle 3\rangle) \to BS^3 \to K(\mathbb{Z},4)$



Kudo $\Rightarrow$ Θi_3 are permanent cycles if $\Theta \in \mathcal{A}_2$ $\Theta \neq 1$

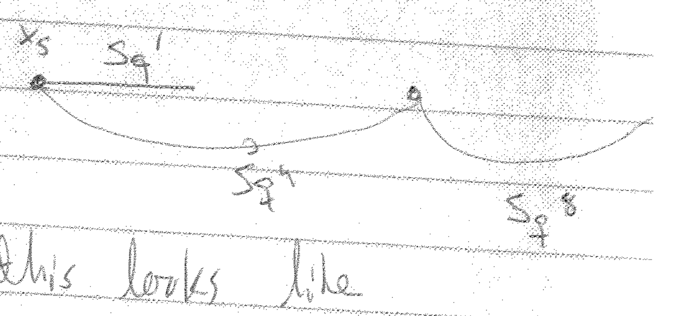
$\Theta = Sq^{2^n, 2^{n-1}, \dots}$

$$H^*(B(S^3\langle 3\rangle)) \cong \mathbb{F}_2[x_5, Sq^1 x_5, Sq^4 x_5, Sq^{8,4} x_5, \dots]$$

$\Updownarrow$ $\Updownarrow$

$Sq^2 i_3$. i_3^2

$QH^*(B(S^3\langle 3\rangle))$



except in low dim this looks like

$\Sigma F(1)$.

(3) For many 1-connected cpt Lie gps they computed $H^*(B(G\langle 3\rangle))$ as an algebra.
It ~~w~~ always look like

$A \otimes B$ $\quad A \subseteq H^* K(\mathbb{Z},3))$

$B \Leftarrow H^*(BG)$.

$(G\langle 3\rangle^{\wedge}_p\ (BG)\langle 4\rangle^{\wedge}_p, e)$ is always p-Noetherian.

<u>Part I:</u> $H^*(BX)$ is finitely generated as an algebra / A_p.

We study fibrations

$$K(A,n) \to E \to X \quad \text{where}$$

1) ~~B~~ E, X are 1-connected.

2) A is an abelian grp of f.t. type.

3) $H^*(X)$ is noetherian.

<u>Thm</u> $H^*(E)$ is f.g. as an algebra / A_p.

<u>Notes:</u> • Not true without assumption 3)

for example

$$K(\mathbb{Z},2) \to \Sigma K(\mathbb{Z},2) \longrightarrow K(\mathbb{Z},3) \vee K(\mathbb{Z},3) \longrightarrow K(\mathbb{Z},3)$$

• Could be true replacing $K(A,n)$ by any f.t. postnikov system

• Might even be true replacing $K(A,n)$ by any space with $H^*(\ )$ f.g. as an alg over A_p

Step 1:

pf of thm: The Serre s.s. collapses at some f.t. stage

H^*X is noetherian, the ascending chain of id of those elements hit by a differential d_k $k \le r$ stabilize

So by Kudo there is a subalgebra $G^* \subseteq H^*(K(A,n))$ generated by "large" Steenrod operatio on the fundamental classes.

$d_{N+|\theta|}\, \theta\, i_n = 0$.

So $H^*(K(A,n)) = G^* \otimes F^*$ as algebras

& f.g. as an algebra

Here G^* but not F^* is stable under Steenrod operations.

In fact $QG^* \subseteq \underbrace{Q H^*(K(A,n))}_{\text{f.g. in } \mathcal{U}}$

So QG^* is f.g. in $\mathcal{U}$ since locally Noetherian.

Consider the quotient s.s.

$$\overline{E}_2 = H^*(X) \otimes F^*.$$

By the Evans-Dwyer-Wilkerson argument this s.s. collapses.

Moreover $\forall r$ $E_r \overset{\text{as algebras}}{=} G^* \otimes \overline{E}_r$ and therefore collapses also on the same page.

By induction
$a \otimes g \in E_r \otimes G$. Then $d_r(a \otimes g) = d_r(a) \otimes g$

So $\bar{E}_{r+1} = E_{r+1} \otimes G^*$

Step 2: The cohomology ~~of~~ $H^*(E)$ is f.g. as an ~~module~~ algebra over G^*.

Pf of
From step 1 we know $\bar{E}_\infty$ is f.g. as a module over $H^*(X) \otimes \mathbb{F}_p[z_1^{p^{n_1}}, \dots, z_s^{p^{n_s}}]$
z_i generators, polynomial, of F^* $z_i^{p^{n_i}}$ permanent cycle
(this is one-variable argument).
$\Rightarrow$ E_∞ is f.g. as a module over $H^*(X) \otimes \mathbb{F}_p[z_1^{p^{n_1}}, \dots, z_s^{p^{n_s}}]$
$\otimes G^*$

$\Rightarrow H^*(E)$ — " —

$\Rightarrow H^*E$ is finitely generated as an algebra over A_p.
(and generators for $H^*(X)$ and $z_i^{p^{n_i}}$

Step 3: H^*E is an algebra

Warning: $G^* = H^*(K(2,3))$ $B^* = H^*(K(2,3))$ with trivial action of A_p. B^* is f.g. as modules over G^*.
the problem we need compatibility between action of A_p and of G^*.

Here G^* acts on H^*E as follows.

$$\begin{tikzcd} H^*E \arrow[r, "i^*"] \arrow[dr] & H^*(K(A,n)) \\ & \cup \\ & \mathrm{Im}\, i^* \\ & \cup \\ & G^* \arrow[uuul, bend left, "s."] \end{tikzcd}$$

G^* free alg

s section of alg. not necc. of alg over A_p.

g in G^* acts via this section.

$$g \cdot x = s(g)x.$$

for $\theta \in A_p$ could have $(\theta g) \cdot x = s(\theta g) \cdot x$

$$\neq \theta\, s(g)x$$

perhaps

But we have map $p: H^*E \xrightarrow{i^*} H^*(K(A,n))$

$$\cong$$

of algebras. $G^* \otimes F^* \longrightarrow G^*$

Choose generators $1 = b_1, b_2, \ldots, b_m$ of H^*E

as a G^*-algebra.

<u>Claim</u>: $b_1, \ldots b_m, s(g_1), \ldots, s(g_k)$

generate H^*E/A_p if g_i gen G^*/A_p.

idea: the only problem is to write $s(\Theta g_i)$ in terms of $s(g_i)$'s.

If $1, c_1, \ldots, c_t$ are module generators for H^*E over $H^*X \otimes \mathbb{F}_p[z, \ldots] \otimes G^*$

we can assume $c_i \in \ker p$.

Consider ~~[illegible]~~ $s(\Theta g) - \Theta s(g)$ so

$$p\big(s\Theta(g) - \Theta s(g)\big) = \Theta g - \Theta g = 0.$$

Reduced to looking at things in the kernel of p

$$\xi = \lambda_0 + \lambda_1 c_1 + \ldots + \lambda_t c_t \qquad \lambda_i \in G^*$$

By induction can handle $\lambda_1, \ldots, \lambda_t$

$$p(\xi) = 0 = \lambda_0 + \lambda p(c_1) + \ldots + \lambda_t p(c_t)$$

so $\lambda_0 = -(\lambda_1 p(c_1) + \ldots \lambda_t p(c_t))$.