Cohomology of Kac-Moody Groups (Carlos Broto)

Thm G Kac-Moody group $\Rightarrow H^*(BG;\mathbb{F}_p)$ is a Noetherian algebra.

$F \leq G$ $H^*(BF;\mathbb{F}_p) \longleftarrow H^*(BG;\mathbb{F}_p)$ f.g. $H^*(BG;\mathbb{F}_p)$-module
Kac-Moody subgp.

Fix prime p.

Def: $\mathcal{X}$ class of cpt gp. K, $\mathcal{X} \ni K$
if there exists a f.t. K-CW-cpx X with
i) The isotropy gps of X belong to $\mathcal{X}$
ii) $\forall$ ft p-gps $\pi \leq K$ X^π is p-acyclic

X satisfying those conditions is called a K p-acyclic cpx.

A GCM $l \times l$

$G(A)$ group associated to the integrable algebra $g'(A)$.

$(G(A), B_+, N, S)$

S system of fundamental reflections $\{\bar{r}_1, \ldots, \bar{r}_l\}$.
$\bar{r}_i \in G(A)$.

$\overline{W}(A) = \langle \bar{r}_1, \ldots, \bar{r}_l \rangle$ $W(A) = \langle r_1, \ldots, r_l \rangle$.

H corresponds $h' := h \cap g'$.

U_+ U_- corresponds Δ_+ and Δ_-

B_+ B_- corresponds to $h' +$ (gen by Δ_+) and $h +$ (gen by Δ_-)

Bruhat decomp: $G(A) = \coprod_{w \in W} B_+ \overline{w} B_+$

Parabolic subgps $I \subseteq S = \{\overline{r}_1, \dots, \overline{r}_\ell\}$.

$\overline{W}_I$ subgp gen by I $\rightsquigarrow$ W_I gen by r_i if $\overline{r}_i \in I$

$P_I = \langle B_+, I \rangle$

Topological Tits building:

$\mathcal{C}$ = category of proper subsets of S.

$$\mathcal{C} \longrightarrow \text{Spaces}.$$

$$I \longrightarrow G(A)/P_I.$$

[S. Mitchell] W infinite $\Rightarrow$ $\operatorname{hocolim}_{\mathcal{C}} F \simeq *$.

$$\operatorname{hocolim}_{\mathcal{C}} F = \left(\Delta^\ell \times G(A)/B_+\right)/\sim$$

$K(A)/T$ has cell decomposition

$$= \coprod_{w \in W} B_+ \overline{w} B_+ / B_+ .$$

Eg. $\ell = 2$

$$\begin{tikzcd} \{1\} \arrow[dr] & \\ & \{1,2\} \\ \{2\} \arrow[ur] & \end{tikzcd}$$

$$\begin{array}{c} K/H_1 \\ | \\ K/T \\ | \\ K/H_2 \end{array} \rightsquigarrow \begin{tikzcd} BT \arrow[r] \arrow[d] & BH_1 \arrow[d] \\ BH_2 \arrow[r] & BK \end{tikzcd}$$

$\mathcal{C}_f$ = category of subsets $\{1, \ldots, \ell\}$.
s.t. the corresponding parabolic P_I is compact

$F : \mathcal{C}_f \to \text{Spaces} \quad F(I) = K/P_I \quad X_K = \operatorname{hocolim}_{\mathcal{C}_f} ?$

$$X_K = (K/_T \times B\mathcal{C}_f)/\sim \quad (gT, x) \sim (hT, y)$$
$$\Leftrightarrow \quad x = y \in \mathring{\Delta}_I$$
$$gP_I = hP_I.$$

Note since X_K contractible

$$\begin{array}{ccc} (X_K)_{hK} & = \operatorname{hocolim}_{\mathcal{C}_f} & (K/P_I)_{hK} \\ \wr\vert & & \wr\vert \\ BK & & \operatorname{hocolim}_{\mathcal{C}_f} BP_I. \end{array}$$

$\ell(w) = \min\{k \mid w = r_{i_1} \ldots r_{i_k}\}$.

Filtration of $K/_T$

$$F_k(K/T) = \coprod_{\substack{w \in W \\ \ell(w) \leq k}} BwB/_B$$

$$F_k(X_K) = (F_k(K/T) \times B\mathcal{C}_f)/\sim.$$

$$F_k(X_k)/_{F_{k-1}(X_k)} = \bigvee_{\ell(w)} \rho(\underbrace{\overline{BwB}/_B \times BC_f})$$

acyclic — proving this is a bit technical

where

$$\rho : F_k(K/T) \times BC_f \twoheadrightarrow F_k(X_k).$$

Showing $\rho(\overline{BwB}/_B \times BC_f)$ is acyclic hence finishes the pf.

Isotropy spectral sequence for G-space X

Leray spectral seq for $X \times_G EG \to X/G$

$$E_1 = \prod_{\sigma \text{ cells}} H^*(H_\sigma; \mathbb{F}_p) \Rightarrow H^*(BG; \mathbb{F}_p)$$

fit. # cells.

noetherian

$\cup\!\!\mid$

K f.g. subalgebra

If we can do this

Get s.s. of K-modules $\Rightarrow H^*(BG; \mathbb{F}_p)$

noetherian

Need to find K.

$\boxed{H^*(\ ) = H^*(\ ; \mathbb{F}_p)}$

$A_p(G)$ category of non-triv elt ab subgps

Have Quillen map $q_G\ H^*(BG) \to \lim\limits_{A_p(G)}{}^{0} H^*(BE)$

(5)

q_G ~~MAA~~ F-iso for G-cpt by Quillens thm

In general.

q_G is F-iso

$\Longleftrightarrow$ V elt ab $\Leftrightarrow$ $\mathrm{Hom}_{\mathcal{U}}(\lim^0, H^*V)$

$\xrightarrow{\cong} \mathrm{Hom}_{\mathcal{U}}(H^*(BG, H^*V)$.

$\mathcal{U}$ = cat of unstable modules over the Steenrod alg.

$\mathcal{K}$ = " — alg — " —

Lannes' T-functor.

V elt ab $T_V : \mathcal{U} \to \mathcal{U}$. left adj to

$- \otimes_{\mathbb{F}_p} H^*(V)$

~~NAAAAA~~ $\mathrm{Hom}_{\mathcal{K}}(H^*(BG), H^*V)$

$\cong$

$\mathrm{Hom}_{\mathcal{K}}(T_V H^*BG, \mathbb{F}_p)$

$\cong$ — next page

$G \in \mathcal{K}, *$

<u>Thm</u> X finite G-CW-cpx then ~~$T_V H^*(X)$~~

$$T_V H^*_G(X) \cong \prod_{\rho \in \mathrm{Rep}(V,G)} H^*_{C_G(\rho)}(X^\rho)$$

In particular $T_V H^*(BG) \cong \prod_{\mathrm{Rep}(V,G)} H^*(BC_G(\rho))$.

$$\cong \mathrm{Hom}_K\Big(\prod_{\mathrm{Rep}(V,G)} H^*(BC_G(\rho)),\ \mathbb{F}_p\Big) = \mathrm{Rep}(V,G)$$

<u>Note:</u> By the assumptions on X, $\mathrm{Rep}(V,G)$ is finite.

<u>Words about proof of Thm p.5.:</u>

- By induction enough to prove for G/H.

$$T_V H^*_G(G/H) = T_V(H^*(BH)) \overset{\text{Lannes}}{\cong} \prod_{\rho\in\mathrm{Rep}(V,H)} H^*(C_H(\rho(V)))$$

$$EG \times_{C_G(\rho)} (G/H)^\rho \cong \coprod_{i=1}^{r} EG \times_{C_G(\rho)} C_G(\rho)/C_G(\rho)\cap g_i H$$

As $C_G(\rho)$-set

- $(G/H)^\rho \cong \coprod_{i=1}^{r} EG \times_{C_H(c_{g_i}\circ\rho)} \{pt\}$.

where g_i coset reps of G/H^ρ i.e. factorizations

$$\begin{tikzcd} & H \arrow[dr, hook] & \\ V \arrow[ur, "\exists"] \arrow[rr] & & G \end{tikzcd}$$

$$\prod_{\rho} \sim\sim \ \_\_\_\_ \ \_\_\_\_$$

END

Now to finish the pf of of char's K. • Prove Q in noetherian unstable ~~gen~~ in ~~noetherian~~ End V-set.

• Use this to lift to gen in $H^*(BG)$ (need to make ~~...~~ closed under Steenrod operation.)