MY RESEARCH

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The study of the interplay between combinatorics, geometry and algebra goes back to antiquity, with regular convex polyhedra already being carved out of stone in the Neolithic period and often conflated with religious mysticism. Contemporary study of combinatorial structures is often motivated by the importance of optimization in modern economy, as well as the ubiquity of combinatorial problems in pure mathematics. It has also, however, become a vital subject in itself.

A popular and central subject of modern combinatorics, polytope theory, experienced its first serious developments by Euler 1758, and later Legendre and Cauchy in the late 18th century and early 19th century. With the dawn of the 19th century, meticulous study of polytopes gained traction. Combinatorialists such as Sommerville were interested in enumerative problems, Schläfli studied the geometric properties of polytopes (including highly symmetric polytopes) and Poincaré introduced topologists to homology and simplicial homology, generating interest in triangulated manifolds and other combinatorial decompositions of topological spaces.

Modern combinatorialists around McMullen and Stanley discovered the relation between enumerative questions on polytopes and toric algebraic geometry, and commutative algebra. Meanwhile, the polyhedral study of 3-manifolds is one of the pillars of the geometrization program of Thurston. This research, together with enumerative problems in algebraic geometry, also created the field of combinatorial topology, where special decompositions of topological and algebraic spaces are studied and analyzed with combinatorial methods.

Let me illustrate this with an example:

1. Counting faces

One of the most basic problems that almost every combinatorialist asks is to count things. And when you ask a combinatorial topologist, they usually want to count things associated to topological spaces.

More precisely: many topological spaces can be triangulated, that is, there are simplicial complexes that encode them. A tetrahedron, for instance, triangulates a 3-dimensional ball. The boundary of that tetrahedron triangulates the 2-dimensional sphere. But so does the boundary of the icosahedron. Now we can count the number of faces that each of these triangulations have:

In the former case, we get one empty face, four vertices, six edges, and four triangles. Nice and symmetric (well, almost). This is usually recorded in a vector called the f-vector recording the number of i-dimensional faces f_i, which reads in this case as

$$(1, 4, 6, 4).$$

In the latter, we get one empty face, twelve vertices, thirty edges, and twenty triangles (where it gets its name), and hence an f-vector

$$(1, 12, 30, 20).$$

This is no longer so nice and symmetric. However, Sommerville 1927 had the idea to define another vector

$$h_k \ := \ \sum_{i=0}^{k}(-1)^{k-i}\binom{d-i}{k-i}f_{i-1}.$$

And now, there is a small miracle: he then established[1] that for a simplicial sphere of dimension $(d-1)$, we have

$$h_k \ = \ h_{d-k}.$$

These are the so-called Dehn-Sommerville relations. This means in particular that there are nontrivial linear relations between the face numbers of simplicial polytopes, and that everything is defined from the first half of the entries. So, McMullen (1971) had the idea to consider another vector:

$$g_k \ := h_k - h_{k-1} \text{ for } k \leq \frac{d}{2}$$

He formulated the following conjecture

Conjecture 1.1. *A vector of d integers is the f-vector of a simplicial sphere Σ if and only if the associated g-vector is an M-sequence, that is, there is a quotient Q of a polynomial ring $\mathbb{R}[\mathbf{x}]$ by a homogeneous ideal so that*

$$g_i(\Sigma) \ = \ \dim Q^i.$$

At this point, Stanley entered. He realized that there is at least always a ring that encodes the h-vector.

2. Rings

If Δ is an abstract simplicial complex defined on the groundset $[n] := \{1, \cdots, n\}$, let $I_\Delta := \langle \mathbf{x}^{\mathbf{a}} : \text{ supp } (\mathbf{a}) \notin \Delta\rangle$ denote the nonface ideal in $\mathbb{R}[\mathbf{x}]$, where $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \cdots, x_n]$. Let $\mathbb{R}^*[\Delta] := \mathbb{R}[\mathbf{x}]/I_\Delta$ denote the **face ring** of Δ. Now, we pick a sufficient number of linear forms to make sure the quotient is finite dimensional:

The reduced face ring with respect to such a system Θ is

$$\mathscr{A}^*(\Delta) \ := \ \mathbb{R}^*[\Delta]/\Theta\mathbb{R}^*[\Delta].$$

Stanley observed, based on a theorem of Reisner (1976):

Theorem 2.1. *For a triangulated sphere Σ of dimension $(d-1)$,*

$$h_i(\Sigma) \ = \ \dim \mathscr{A}^i(\Sigma).$$

3. Stanley and Lefschetz

Here, Stanley observed that McMullen's conjecture is true if there existed an ℓ in $\mathscr{A}^1[\Sigma]$ so that

$$\mathscr{A}^i(\Sigma) \ \xrightarrow{\cdot \ell} \ \mathscr{A}^{i+1}(\Sigma)$$

is injective for $i \leq \frac{d}{2} - 1$, or stronger if

$$\mathscr{A}^k(\Sigma) \ \xrightarrow{\cdot \ell^{d-2k}} \ \mathscr{A}^{d-k}(\Sigma).$$

is an isomorphism for every $k \leq \frac{d}{2}$. The former is known as the **weak Lefschetz property**, the latter as the **hard Lefschetz property** in algebraic geometry. He needed this for *some* geometric realization of the simplicial complex, that is, some choice.

[1] He stated this for boundaries of simplicial polytopes, but the proof works for simplicial spheres

And amazingly, Stanley then observed that the hard Lefschetz property is actually true for spheres that arise as boundaries of simplicial polytopes (with respect to their given geometric realization) using deep results in algebraic geometry. Specifically, if Σ is realized as the boundary of a polytope, then the class of a convex function acts as the desired Lefschetz element. But the general case remained open, and only recently, I was able to prove the hard Lefschetz theorem the necessary generality. Indeed, this was done by mostly combinatorial methods, no longer relying on algebraic geometry in any way.