Higher algebra

Lars Hesselholt

Let $f\colon X \to \operatorname{Spec}(\mathbb{Z})$ be a proper morphism of schemes with X regular, geometrically connected, and of dimension d. If $d = 1$, then X is either the prime spectrum of a ring of integers in a number field or a smooth and proper curve over a finite field. Counting rational points on X is organized into the Hasse–Weil zeta function, which is defined, for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > d$, by the Euler product

$$\zeta(X,s) = \prod_{x \in |X|} (1 - \operatorname{card}(k(x))^{-s})^{-1}.$$

Here $|X|$ is the set of closed points of X and $k(x)$ is the (finite) residue field at x. It is expected that $\zeta(X,s)$ admits an extension to a meromorphic function on $\mathbb{C}$. Serre, following Riemann, defined an archimedean Euler factor $\zeta_\infty(X,s)$ determined by $X \otimes_{\mathbb{Z}} \mathbb{R}$ with its real Hodge structure. The completed zeta function $\zeta(\bar{X},s) = \zeta(X,s) \cdot \zeta_\infty(X,s)$ is expected to satisfy a functional equation of the form

$$\zeta(\bar{X}, d-s) = A \cdot B^s \cdot \zeta(\bar{X},s)$$

as well as the generalized Riemann hypothesis: if $s \in \mathbb{C}$ is a zero (resp. a pole) of $\zeta(\bar{X},s)$, then

$$\langle s,s \rangle = 2\operatorname{Re}(s)$$

is an odd (resp. even) integer $0 \leq v \leq 2d$. One would like to have a cohomological interpretation of the completed zeta function in such a way that the functional equation and the Riemann hypothesis follow from Poincaré duality and the existence of a Hodge $*$-operator, respectively.[1] Deninger has given a detailed proposal as to the form that such a cohomology theory should take. In particular, it should assign to $f\colon X \to \operatorname{Spec}(\mathbb{Z})$ a graded $\mathbb{C}$-algebra equipped with a Frobenius flow

$$\operatorname{Fr}_t^* = t^\theta$$

through graded $\mathbb{C}$-algebra homomorphisms.[2] The dimension of the graded $\mathbb{C}$-algebra must necessarily be infinite, because the completed zeta function typically has infinitely many zeros or poles.

Suppose that $f\colon X \to \operatorname{Spec}(\mathbb{Z})$ is supported at a single prime, and hence, admits a factorization

$$X \xrightarrow{\ h\ } \operatorname{Spec}(\mathbb{F}_p) \xrightarrow{\ i\ } \operatorname{Spec}(\mathbb{Z})$$

with h smooth and proper. In this case, the (completed) zeta function $\zeta(X,s)$ is periodic with period $2\pi i/\log p$, or equivalently, can be written as $\zeta(X,s) = Z(X,p^{-s})$ for some other function $Z(X,t)$. The Grothendieck school found a cohomological interpretation of $Z(X,t)$ by a finite dimensional cohomology theory, and used this to show that $\zeta(X,s)$ indeed has the properties enumerated above.[3] This is possible, because $Z(X,t)$ turns out to be a rational function with integer coefficients. In recent work, I have instead given a cohomological interpretation of the function $\zeta(X,s)$ itself in the form envisioned by Deninger. The relevant cohomology theory is the higher algebra analogue of de Rham cohomology, which I will now describe.

The seeds for higher algebra were planted by Boardman and Vogt in the early seventies, but the theory was only fully developed much later by Joyal and Lurie. The core idea is to replace sets

[1] The questions of whether the functional equation and the Riemann hypothesis hold are thus decoupled from the question of whether a meromorphic extension of the zeta function exists.

[2] Equivalently, the infinitesimal generator θ should be a graded derivation.

[3] Grothendieck and Deligne both received a Fields medal for their work.

by spaces.[4] So the notion of equality “$a = b$” is replaced by the weaker notion that “a and b are points in a common contractible space.” This has numerous consequences, chiefly among which are that properties (such as commutativity) get replaced by structures (such as an $\mathbb{E}_\infty$-algebra structure). For example, in algebra, the free commutative monoid on one generator is the monoid $\mathbb{N}$ of natural numbers under addition, and the free commutative group on one generator is the group $\mathbb{Z}$ of integers under addition. By contrast, in higher algebra, the free commutative monoid on one generator is the (cocartesian symmetric monoidal) category $\mathrm{Fin}^{\simeq}$ of finite sets and isomorphisms, and the free commutative group on one generator is the sphere spectrum $\mathbb{S}$.

$$\begin{array}{ccccc} \mathbb{N} & \rightsquigarrow & \mathbb{Z} & \qquad & \text{algebra} \\ | & & | & & | \\ \mathrm{Fin}^{\simeq} & \rightsquigarrow & \mathbb{S} & & \text{higher algebra} \end{array}$$

In algebra, the commutative group $\mathbb{Z}$ admits a unique structure of commutative ring, and this ring is initial among commutative rings. Similarly, in higher algebra, the commutative group $\mathbb{S}$ admits a unique, up to contractible choice, structure of commutative ring,[5] and this ring is initial among commutative rings.

We will now place ourselves in higher algebra, where we let k be a commutative ring, and A a commutative k-algebra. We can form the tensor power $A^{\otimes_k X}$ for every space X, and letting X be the unit circle $S^1 \subset \mathbb{C}$, we obtain (not obviously) the higher algebra analogue of differential forms,

$$\mathrm{HH}(A/k) = A^{\otimes_k S^1}.$$

The standard action of the circle group $U(1)$ on $S^1 \subset \mathbb{C}$ induces, by functoriality, a $U(1)$-action on $\mathrm{HH}(A/k)$, and the higher algebra analogue of de Rham cohomology is the Tate construction

$$\mathrm{HP}(A/k) = \mathrm{HH}(A/k)^{tU(1)}$$

of this action. This is the periodic cyclic homology of Connes and Tsygan. It can be defined, more generally, for A a k-linear ∞-category.

Now, let $h\colon X \to \mathrm{Spec}(\mathbb{F}_p)$ be a smooth and proper morphism and let $\mathrm{Mod}^{\omega}_{\mathscr{O}_X}$ be the ∞-category of perfect $\mathscr{O}_X$-modules. We consider the periodic topological cyclic homology defined by

$$\mathrm{TP}(X) = \mathrm{HP}(\mathrm{Mod}^{\omega}_{\mathscr{O}_X}/\mathbb{S}).$$

Its homotopy groups $\mathrm{TP}_*(X)$ form a graded algebra over the ring $\mathbb{Z}_p = A_{\mathrm{inf}}(\mathbb{F}_p)$ of p-adic integers, and the geometric Frobenius $\mathrm{Fr}_p\colon X \to X$ induces an endomorphism of this graded $\mathbb{Z}_p$-algebra. We now commit the crime of choosing an embedding of rings $\iota\colon \mathbb{Z}_p \to \mathbb{C}$ in order to form the graded $\mathbb{C}$-algebra $\mathrm{TP}_*(X) \otimes_{\mathbb{Z}_p} \mathbb{C}$. For any such choice, I show that there is a cohomological interpretation

$$\zeta(X,s) = \frac{\det_\infty(\frac{1}{2\pi}(s\cdot \mathrm{id} - \theta) \mid \mathrm{TP}_{\mathrm{od}}(X) \otimes_{\mathbb{Z}_p} \mathbb{C})}{\det_\infty(\frac{1}{2\pi}(s\cdot \mathrm{id} - \theta) \mid \mathrm{TP}_{\mathrm{ev}}(X) \otimes_{\mathbb{Z}_p} \mathbb{C})}$$

of the form envisioned by Deninger. Recall that the operator θ should be the infinitesimal generator of a Frobenius flow Fr_t^*. However, we only have the single value $t = p$ available. But, on the one hand, the equation $\mathrm{Fr}_p^* = p^\theta$ determines θ, up to a choice of branch of the logarithm with base p, and, on the other hand, the graded $\mathbb{C}$-vector space $\mathrm{TP}_*(X) \otimes_{\mathbb{Z}_p} \mathbb{C}$ is 2-periodic. Therefore, we can choose to use all branches of the logarithm to define θ, and if we do so, then the formula holds. The regularized determinant $\det_\infty$ is an extension of the usual determinant to (some) $\mathbb{C}$-linear endomorphisms of infinite-dimensional $\mathbb{C}$-vector spaces.[6] Antieau–Mathew–Nikolaus have proved a Poincaré duality

[4] Here a “space” means a topological space, but only as far as homotopy theory is concerned. So for Joyal and Lurie, a “space” means a Kan simplicial set.

[5] In higher algebra, “commutative ring” means an $\mathbb{E}_\infty$-algebra in the symmetric monoidal ∞-category of spectra $\mathrm{Sp}^{\otimes}$.

[6] These are $\mathbb{C}$-vector spaces is the algebraic sense; they are not topological $\mathbb{C}$-vector spaces.

theorem for TP from which the functional equation for $\zeta(X,s)$ ensues. One would like to also define a conjugate linear Hodge $*$-operator on $\mathrm{TP}_*(X)\otimes_{\mathbb{Z}_p}\mathbb{C}$ and prove the Riemann hypothesis. However, the only ring automorphism of $\mathbb{Z}_p$ is the identity map, so the complex conjugation on $\mathrm{TP}_*(X)\otimes_{\mathbb{Z}_p}\mathbb{C}$ is not visible on $\mathrm{TP}_*(X)$. It is not unreasonable to hope that p-adic Hodge theory can be extended to address this problem.

One hopes to extend the results above to general $f\colon X\to\mathrm{Spec}(\mathbb{Z})$. There are formidable difficulties in doing so. However, the use of the geometric Frobenius to define θ is not among them. For it turns out to be possible to reconstruct the operator Fr_p^* without using the morphism $\mathrm{Fr}_p\colon X\to X$, as I will now explain. There is a "cyclotomic" Frobenius map[7]

$$\mathrm{TP}_*(X)\otimes_{\mathbb{Z}_p}\mathbb{C}\xrightarrow{\varphi_p}\mathrm{TP}_*(X)\otimes_{\mathbb{Z}_p}\mathbb{C},$$

which was discovered by Bökstedt–Hsiang–Madsen. This map exists only in higher algebra and not in algebra. In addition, Bhatt–Morrow–Scholze have constructed a "weight" filtration[8]

$$\cdots\subset\mathrm{Fil}^{w+1}\mathrm{TP}(X)\subset\mathrm{Fil}^{w}\mathrm{TP}(X)\subset\mathrm{Fil}^{w-1}\mathrm{TP}(X)\subset\cdots\subset\mathrm{TP}(X)$$

for any scheme X. Now, if $h\colon X\to\mathrm{Spec}(\mathbb{F}_p)$ is smooth and proper, then the equality

$$\mathrm{Fr}_p^*=p^w\varphi_p$$

holds on the wth graded piece of the weight filtration of $\mathrm{TP}_*(X)\otimes_{\mathbb{Z}_p}\mathbb{C}$, and hence, we can use the left-hand side of this equation as our definition of the operator Fr_p^*. In the case $X=\mathrm{Spec}(\mathbb{Z})$, the resulting operator acts on $\mathrm{TP}_{\mathrm{od}}(X)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ with the correct eigenvalues to account for the pth Euler factor in the Riemann zeta function. However, the $\mathbb{Q}_p$-vector space that this operator acts on is of the from $V=\bigoplus_{m\in\mathbb{Z}}((\prod_{n\in\mathbb{N}}\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)$, but to obtain a cohomological interpretation of the Riemann zeta function, we should instead have an operator with the same eigenvalues acting on a $\mathbb{C}$-vector space of the form $W=\bigoplus_{m\in\mathbb{Z}}\mathbb{C}$.

[7] It is not necessary to extend scalars to $\mathbb{C}$; the map φ_p exists after extending scalars to $\mathbb{Q}_p$.

[8] The graded pieces of the weight filtration are the "new cohomology theories" mentioned in the citation for Scholze's Fields medal in 2018.