Real functional analysis

Q. Can condensed sets, being built from profinite sets, possibly be a good language to talk about "real" stuff?

$$\underset{\substack{\text{profinite} \\ \text{(or extr. disc.)}}}{S} \twoheadrightarrow \underset{\substack{\cap \\ \text{Cond Set}}}{[0,1]} = S/R.$$

$$[0,1] = \left\{ \begin{array}{c} \text{decimal expansions} \\ 0.395\ldots \end{array} \right\} \Big/ \cong .$$

$$\left(0.0999\ldots = 0.1000\ldots \right.$$

$$\prod_{n \geq 1} \{0,1,\ldots,9\}. \quad \underline{\text{profinite}}.$$

<u>More fancy:</u> Any $x \in \mathbb{R}$ can be written

as $\displaystyle\sum_{n \gg -\infty} a_n \cdot \frac{1}{10^n}$ $\quad a_n \in \mathbb{Z}$ (bounded, or not grow too fast).

$$\mathbb{Z}((T))_{conv} = \left\{ f(T) = \sum_{n \gg -\infty} a_n T^n \in \mathbb{Z}((T)) \;\middle|\; \begin{array}{l} f \text{ converges on} \\ \{0 < |T| < 1\} \end{array} \right\}.$$

$$\begin{array}{ccc} \mathbb{Z}((T))_{conv} & & T \\ \downarrow & & \downarrow \\ \mathbb{R}. & & \frac{1}{10}. \end{array}$$

$$\mathbb{R} = \mathbb{Z}((T))_{conv} / \left(T - \frac{1}{10}\right).$$

A. Yes, it works. But it takes some work.

Goal: Define an analogue of category of solid $\mathbb{Q}_p$-vector space for $\mathbb{R}$.

<u>Recall:</u> $\mathbb{R}^{\blacksquare} = 0$, so

$$\mathrm{Mod}_{\mathbb{R}}(\mathrm{Solid}_{\mathbb{Z}}) = 0.$$

The formalism should be similar:

full subcat. of condensed $\mathbb{R}$-vector spaces specified by a functor.

$$L : \mathrm{Cond}(\mathbb{R})^{op} \longrightarrow \mathrm{Cond}(\mathbb{R})$$
$$\mathbb{R}[S] \longmapsto L(\mathbb{R}[S])$$

satisfying Dustin's axioms.

<u>What should $L(\mathbb{R}[S])$ be?</u>

Some descriptions of $\mathbb{Q}_p[S]^{\blacksquare}$: $S = \lim_i S_i$

1) $\left(\varprojlim_i \mathbb{Z}_p[S_i]\right)\left[\frac{1}{p}\right]$.

2) $\underline{\mathrm{Hom}}(C(S,\mathbb{Z}_p),\mathbb{Z}_p)[\frac{1}{p}]$

$\quad\quad \|$

$\underline{\mathrm{Hom}}(C(S,\mathbb{Q}_p),\mathbb{Q}_p)$ dual of continuous functions.

$\quad\quad \|$

$\mathcal{M}(S,\mathbb{Q}_p)$ measures on S.

If $V \in \mathrm{Solid}_{\mathbb{Q}_p}$,

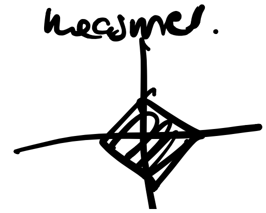
$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & V \\ \mathcal{M}(S,\mathbb{Q}_p) \arrow[ur, dotted, "\exists !"'] & \end{tikzcd}$$

$s \in S \mapsto \delta_s \in \mathcal{M}(S,\mathbb{Q}_p)$ Dirac measures

$\rightsquigarrow$ $\forall \mu \in \mathcal{M}(S,\mathbb{Q}_p)$, can define $\int f \mu \in V$.

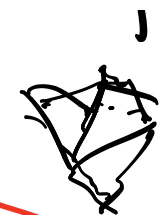
$\rightsquigarrow$ First guess for $L(\mathbb{R}[S])$:

Take $\mathcal{M}(S,\mathbb{R})$. signed Radon measures.



$\quad\quad \|$

$\mathrm{Hom}(C(S,\mathbb{R}),\mathbb{R})$



Compact.

Prop'n. $\mathcal{M}(S,\mathbb{R}) = \bigcup_{c>0} \lim_i \underbrace{\mathbb{R}[S_i]_{\ell^1 \le c}}_{\text{compact convex subsets.}}$,

is a "Smith space".

Def'n. A smith space is a compactly gen. top. $\mathbb{R}$-vector space W s.th.

$$W = \bigcup_{c>0} c\cdot K, \quad K \subseteq W$$

compact convex subset.

symmetric.

If $x, y \in K$, $a, b \in \mathbb{R}$, $|a|+|b| \le 1$

$\Rightarrow ax + by \in K$.

Def'n. A condensed $\mathbb{R}$-vector space V is $\mathcal{M}$-complete if it is quasiseparated and for all profin. sets S, $f : S \to V$,

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\text{Dirac meas.}"'] & V \arrow[d, equal] \\ \mathcal{M}(S,\mathbb{R}) \arrow[r, "\tilde{f}"] & V \end{tikzcd}$$

$\exists$ extension to $\tilde{f} : \mathcal{M}(S,\mathbb{R}) \to V$..

(nec. unique) map of cond. $\mathbb{R}$-v.s.

<u>Prop'n</u>. Any Banach space, in fact any complete locally convex $\mathbb{R}$-v.s. V, is $\mathcal{M}$-complete.

<u>Proof</u>. Quasiseparated easy.

Take $f : S \to V$, $\mu \in \mathcal{M}(S,\mathbb{R})_{\leq 1}$

$$= \lim_i \underbrace{\mathcal{M}(S_i,\mathbb{R})_{\leq 1}}_{= \mathbb{R}[S_i]_{\ell^1 \leq 1}}.$$

Pick any lift

$t_i : S_i \to S$ of

$\pi_i : S \to S_i$.

Can define a net in V param. by i's.

$$v_i = \sum_{s \in S_i} f(t_i(s))\, \mu(\pi_i^{-1}(s)) \in V.$$

want to see that this is a Cauchy net, so pick abs. convex open nbhd $\mathcal{U}$ of 0.

i large $\Rightarrow$ $f(t_i(s)) - f(t_i'(s)) \in \mathcal{U}$

for any other choice $t_i' : S_i \to S$.

$$\sum_{s \in S_i} |\mu(\pi_i^{-1}(s))| \leq 1 \qquad \mathcal{U} \text{ abs. convex.}$$

$$\Rightarrow \sum_{s \in S_i} \left(f(t_i(s)) - f(t_i'(s)) \right) \mu(\pi_i^{-1}(s)) \in \mathcal{U}$$

$\rightsquigarrow$ v_i indep't of choices "up to $\mathcal{U}$", so give net.

V complete $\rightsquigarrow$ get unique limit $v \in V$.

$\rightsquigarrow$ map of underlying $\mathbb{R}$-v.s.

$$\mathcal{M}(S, \mathbb{R}) \longrightarrow V.$$

Check: This is continuous. $\square$.

Prop'n. $V \in \mathrm{Cond}(\mathbb{R})$ is $\mathcal{M}$-complete iff it is a filt. union of Smith spaces.

Proof. Given any $f: S \to V$ profinite

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & V \\ \mathcal{M}(S,\mathbb{R}) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

enough: image of

$$\begin{tikzcd} \mathcal{M}(S,\mathbb{R}) \arrow[r] & V \\ \mathcal{M}(S,\mathbb{R})_{\leq 1} \arrow[u, phantom, "\cup"] \arrow[r, two heads] & K \arrow[u, phantom, "\cup"] \end{tikzcd}$$

is a Smith space.

qs.

compact, i.e. qcqs. — qcqs, i.e. compact Hausdorff. abs. convex.

Then image $= \bigcup_{c>0} cK$ Smith space. $\square$.

Prop'n. (Smith '50s) Smith spaces are anti-equiv. to Banach spaces.

$W \mapsto \underline{\mathrm{Hom}}(W,\mathbb{R})$ $\quad V \mapsto \underline{\mathrm{Hom}}(V,\mathbb{R})$.

Sketch. $C(S,\mathbb{R}) \mapsto \mathcal{M}(S,\mathbb{R})$ ok.

$\mathcal{M}(S,\mathbb{R}) \mapsto C(S,\mathbb{R})$ ok.

In general, any Banach V admits inj. w/ closed image

$$0 \to V \to C(S,\mathbb{R}) \to C(S',\mathbb{R}).$$

$\rightsquigarrow$ reduce to previous case, using Hahn-Banach.

(e.g. $S \twoheadrightarrow V^{*}_{\leq 1}$ $\leftarrow$ compact by Banach-Alaoglu) $\square$.

$\otimes$-Products

Prop'n. For $V, V' \in \mathrm{Cond}(\mathbb{R})$ $\mathcal{M}$-complete, $\exists$ $\mathcal{M}$-complete $V \hat{\otimes} V' \in \mathrm{Cond}(\mathbb{R})$ representing bilinear maps.

($= \mathcal{M}$-completion of $V \otimes_{\mathbb{R}} V'$).

In fact $\{\mathcal{M}\text{-complete } V\} \hookrightarrow \mathrm{Cond}(\mathbb{R})$

has a left adjoint, unique colimit-pres. ext. of $\mathbb{R}[S] \mapsto \mathcal{M}(S,\mathbb{R})$.

<u>Sketch</u>. need to show existence of left adj. on general V.

$$\begin{tikzcd}
\bigoplus_j \mathbb{R}[T_j] \arrow[r] \arrow[d] & \bigoplus_i \mathbb{R}[S_i] \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 \\
\bigoplus_j \mathcal{M}(T_j,\mathbb{R}) \arrow[r] & \bigoplus_i \mathcal{M}(S_i,\mathbb{R}) \arrow[r] & \tilde{V} \arrow[r] \arrow[d] & 0 \\
 & & \tilde{V}^{qsep} &
\end{tikzcd}$$

might not be qs.

$\tilde{V}^{qsep} = \mathcal{M}$-completion of V. $\square$

<u>Classical $\otimes$-products of Banachs:</u>

- proj. $\otimes$- product

$V \otimes_\pi W$: repr. bilinear maps to Banachs.

"ℓ^1-bounded sums of tensors"

- injective $\otimes$-product

$$V \otimes_\varepsilon W:$$

$$C(S,\mathbb{R}) \otimes_\varepsilon C(T,\mathbb{R}) = C(S\times T,\mathbb{R}).$$

"ℓ^∞-bounded sums of tensors".

Prop'n. $V \otimes_\pi W = V \hat{\otimes} W$.

(Computation similar to p-adic case.)

Prop'n. $V, W \leadsto$ dual Smith spaces V^*, W^*.

Then $V^* \hat{\otimes} W^*$ is a Smith space, dual is

$V \otimes_\varepsilon W$ (if one of V, W satisfies "approximation property").

Projective objects in Smith spaces:

S extr. disc. $\to$ $\mathcal{M}(S,\mathbb{R})$ projective.

(Dually: $C(S, \mathbb{R})$ injective.)

<u>Open Question</u>. Is any injective Banach space of this form?

<u>Bad News</u>. If S_1, S_2 extr. disc., infinite,
(Cembranos '80)
then $\mathcal{M}(S_1, \mathbb{R}) \hat{\otimes} \mathcal{M}(S_2, \mathbb{R})$ is <u>never</u>
$$\mathcal{M}(S_1, \mathbb{R}) \hat{\otimes} \mathcal{M}(S_2, \mathbb{R}) = \mathcal{M}(S_1 \times S_2, \mathbb{R})$$
projective again.

So far, we neglected non-qs spaces by fiat. If want nice abelian category, need to allow them.

<u>Bad News</u>. The category of $\mathcal{M}$-complete cond. $\mathbb{R}$-v.s. is <u>not</u> stable under extensions.

In fact there are nonsplit extensions

$$0 \to \mathbb{R} \to \mathcal{E} \to \ell^1(S, \mathbb{R}) \to 0.$$

for say $S = \mathbb{N} \cup \{\infty\}$.

<u>Ribe extension</u>. For finite set S,
('70s).

$$0 \to \mathbb{R} \to \mathcal{E}_S \to \mathbb{R}[S] \to 0$$

$$\mathcal{E}_S = \{((x_s)_{s\in S}, y)\}$$

$$\mathcal{E}_S = \bigcup_{c>0} \Big\{((x_s)_s, y) \Big/ \sum |x_s| \le c, \; \Big|y - \sum \underbrace{x_s \log|x_s|}_{\text{not additive, but locally almost linear}}\Big| \le c\Big\}.$$

$\mathcal{E}_{S, \le c}$

$$\big|x \lg|x| + y\lg|y| - (x+y)\lg|x+y|\big| \le 2(|x|+|y|).$$

$$\mathcal{E}_{S, \leq c} + \mathcal{E}_{S, \leq c} \subseteq \mathcal{E}_{S, \leq 4c}.$$

For infinite S, can define
$\text{"}\lim_i S_i\text{"}$

$$0 \to \mathbb{R} \to \mathcal{E}_S = \bigcup_{c>0} \underbrace{\mathcal{E}_{S, \leq c}}_{\substack{\shortparallel \\ \lim_i \mathcal{E}_{S_i, \leq c}}} \longrightarrow \mathcal{M}(S, \mathbb{R}) \to 0$$

In fact, $\mathrm{Ext}^1(V, \mathbb{R})$ for Banach V.

$$\mathrm{Ext}^1(V,\mathbb{R}) = \{\text{locally almost linear maps } V \to \mathbb{R}\} / \{\text{globally almost linear } V \to \mathbb{R}\}$$

$\cup$

entropy.

$x \longmapsto x \lg |x|$.

non-locally convex ext's of locally

Convex vector spaces.

Def'n. Let $0 < p \leq 1$. A p-**Banach** is a top. $\mathbb{R}$-v.s. V s.th. there exists a p-norm

$$\|\cdot\| : V \to \mathbb{R}_{\geq 0}$$

s.th.
i) $\|v\| = 0 \iff v = 0$
ii) $\|v+w\| \leq \|v\| + \|w\|$
iii) defines top.: $\{ v \mid \|v\| < \varepsilon \} \subset V$ basis of nbhd of 0.
iv) complete.
v) $\|av\| = |a|^p \|v\|$ $\quad a \in \mathbb{R}$ $v \in V$

Example. $\ell^p(\mathbb{N}) = \{ (a_n)_n \mid \sum |a_n|^p < \infty \}$

satisfies usual triangle ineq.

not locally convex for $p < 1$.

Remark. If $p' < p$

$$p\text{-Banach} \to p'\text{-Banach}$$
$$\|\cdot\| \mapsto \|\cdot\|^{p'/p}.$$

Def'n. Quasi Banach $=$ p-Banach for some $p > 0$

Thm (Kalton '80s). An ext. of p-Banachs is p'-Banach for all $p' < p$.

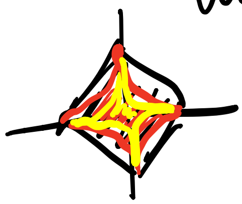
"$< p$-Banachs are stable under extensions".

This suggests following modification of

$$\mathcal{M}(S, \mathbb{R}) = \bigcup_{c>0} \lim_i \mathbb{R}[S_i]_{\ell^q \leq c}:$$

Definition. p-measures $0 < p \leq 1$

$$\mathcal{M}_p(S,\mathbb{R}) = \bigcup_{c>0} \lim_i \mathbb{R}[S_i]_{\ell^p \le c}.$$



$p=1$ "Smith space version of a p-Banach".

$$\mathcal{M}_{<p}(S,\mathbb{R}) = \bigcup_{p'<p} \mathcal{M}_{p'}(S,\mathbb{R}).$$

$\mathbb{R}[S]$ "sums of Dirac measures".

Theorem. $\mathbb{R}[S] \longmapsto \mathcal{M}_{<p}(S,\mathbb{R})$ defines a functor L satisfying Dustin's axioms. In particular:

Def'n. A cond. $\mathbb{R}$-vector space V is *p-liquid* $(0 < p \le 1)$ if the follow. equiv. conditions hold:

1) For all $p' < p$, all $f : S \longrightarrow V$, $\exists!\ f_{p'} : \mathcal{M}_{p'}(S,\mathbb{R}) \longrightarrow V$ extension of f.

2) For all $f: S \longrightarrow V$,

$\exists!\ \tilde{f}_{<p}: \mathcal{M}_{<p}(S, \mathbb{R}) \longrightarrow V$.

3) V can be rewritten as a cokernel of a map

$$\bigoplus_j \mathcal{M}_{<p}(T_j, \mathbb{R}) \longrightarrow \bigoplus_i \mathcal{M}_{<p}(S_i, \mathbb{R}).$$

left adj.

$\rightsquigarrow \quad \mathrm{Liq}_p(\mathbb{R}) \subseteq \mathrm{Cond}(\mathbb{R})$,

$- \otimes_p^{\mathrm{liq}} -$, everything passes to derived categories, ...

p - liquid $\supseteq$ $<p$ - Banachs.

need to prove that for all p-Banachs V, $0 < p' < p \leq 1$.

$R\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{M}_{p'}(S, \mathbb{R}), V)$ ← need to compute this, ... higher want: no higher Ext's.

$$\downarrow 2$$

$$\underline{R\mathrm{Hom}}_R(R[S], V) \longleftarrow \text{conc. in deg. } 0.$$

$$= C(S,V)$$

Breen – Deligne resolution.

$$R[\underbrace{\overbrace{\mathcal{M}_{p'}(S,R)_{\le 1}}^{T}}_{\in \mathrm{CHaus}}] \longrightarrow \mathcal{M}_{p'}(S,R) \longrightarrow 0$$

$R[\text{similar things}] \longrightarrow$

Real Problem. If V p-Banach, $p<1$,

$T \in \mathrm{CHaus}$

$$H^i_{\mathrm{cond}}(T, V) = ???$$

probably nonzero for all i.

expect that one cannot control this.

You really have to resolve by

$\mathbb{R}[T]$'s, with T profinite:

"Explicitly resolve $\mathbb{R}$-v.s. by (free $\mathbb{R}$-v.s. on) profinite sets."

<u>Way out:</u> Write

$$\mathbb{R} = \mathbb{Z}((T))_{\text{conv}} \Big/ (10T - 1).$$

$\mathbb{Q}_p$.

union of profinite things.

define an analogous liquid theory over

$\mathbb{Z}((T))_{\text{conv}}$, then execute the

intended strategy.

Liquid $\mathbb{Q}_p$-vector spaces also make sense.

all locl. $\mathbb{Q}_\ell$-v.s. "gaseous" solid

liquid

$p = 0$ $\quad 0 < p < \infty$ $\quad p = \infty$