Liquid real vector spaces

Reference: lectures 3 – 9 of Analytic.pdf.
(3–6 yesterday, 6–9 today).

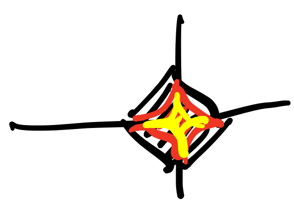
Goal: Define a theory of "liquid" $\mathbb{R}$-v.s., extending complete locally convex $\mathbb{R}$-v.s., but having the same excellent categorical properties as solid $\mathbb{Q}_p$-v.s.
In particular, get nice derived category.

Def'n. $0 < p \leq 1$. p-measures

$$\mathcal{M}_p(S) = \bigcup_{c>0} \lim_i \mathbb{R}[S_i]_{\ell^p \leq c}.$$

$S = \lim_i S_i$

$$\mathcal{M}_{<p}(S) = \bigcup_{p'<p} \mathcal{M}_{p'}(S)$$



$p = 1$:

$p = 1/2$

$p = 1/4$

Remark. Any $\mu \in \mathcal{M}_p(S)$ (*) $p<1$

(Exercise) is a countable sum of Dirac measures.

Thm. $(\mathbb{R}, S \mapsto \mathcal{M}_{<p}(S))$ defines an "analytic ring": satisfies this axiom (*) from Dustin's talk.

Key. If V p-Banach space over $\mathbb{R}$, then

$$R\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{M}_{<p}(S), V) \xrightarrow{\sim} R\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathbb{R}[S], V).$$

Remark. $(\mathbb{R}, S \mapsto \mathcal{M}_p(S))$ does **not** define an analytic ring for any $0<p\leq 1$.

In fact, there are functorial (in S) in general nonsplit extensions.

$$0 \longrightarrow \mathcal{M}_p(S) \longrightarrow \bullet \longrightarrow \mathcal{M}_p(S) \to 0.$$

(can be constructed explicitly using entropy).

$\mathbb{R}[\varepsilon]/\varepsilon^2$-module.

"deformation of $(\mathbb{R}, S \mapsto \mathcal{M}_p(S))$ to $\mathbb{R}[\varepsilon]/\varepsilon^2$."

In fact can deform all the way to $\mathbb{R}[\![\varepsilon]\!]$.

slightly tricky to define.

has passing resemblance to Fontaine's construction of $B_{dR}^+ \cong \mathbb{Q}_p[\![\varepsilon]\!] \twoheadrightarrow \mathbb{C}_p$.

<u>Problem in proving Thm</u>.

need to resolve $\mathbb{R}$-v.s by (free $\mathbb{R}$-v.s. on) profinite sets.

Have to change the base ring.

$$\mathbb{Z}((T))_{conv} \twoheadrightarrow \mathbb{R}.$$

Definition. Let $0<r<1$. Let $\mathbb{Z}((T))_r$ be the condensed ring given by

$$\mathbb{Z}((T))_r = \Big\{ f = \sum_{n\gg-\infty} a_n T^n \in \mathbb{Z}((T)) \Big/ \sum |a_n| r^n < \infty \Big\}.$$

$$= \bigcup_{c>0} \underbrace{\Big\{ f \;\Big|\; \sum |a_n| r^n \le c \Big\}}.$$

Countable union of profinite.

are profinite: each a_n can take only fin. many values, $=0$ if $n \ll 0$.

$f \in \mathbb{Z}((T))_r$ conv. for $0<|T|\le r$ $\quad T \in \mathbb{C}$.

For any finite set S, let

$$\mathbb{Z}((T))_r[S]_{\le c} = \Big\{ \sum_{n,\, s\in S} a_{n,s} T^n [s] \Big/ \sum |a_{n,s}| r^n \le c \Big\}$$

$$\mathcal{M}(S, \mathbb{Z}((T))_r) = \bigcup_{c>0} \varinjlim_i \mathbb{Z}((T))_r[S_i]_{\le c}.$$

Also, set

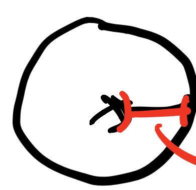
$$\mathbb{Z}((T))_{>r} = \bigcup_{r'>r} \mathbb{Z}((T))_{r'}$$

$$\mathcal{M}(S, \mathbb{Z}((T))_{>r}) = \bigcup_{r'>r} \mathcal{M}(S, \mathbb{Z}((T))_{r'}).$$

For each $0 < t < r$, get map

$$\begin{aligned} \theta_t : \mathbb{Z}((T))_r &\longrightarrow \mathbb{R} \\ T &\longmapsto t. \end{aligned}$$

<u>Prop'n</u>. 1) $\ker(\theta_t) = (f_t)$ principal.



$0<t\le r.$

2) $\exists$ can. isom.

$$\mathcal{M}(S, \mathbb{Z}((T))_r)/(f_t) \cong \mathcal{M}_p(S, \mathbb{R})$$

as cond. $\mathbb{Z}((T))_r/(f_t) \underset{1)}{\cong} \mathbb{R}$ - modules,

where $0<p<1$ s.th. $t^p = r$.

3)

$$0 \to \mathcal{M}(S, \mathbb{Z}((T))_r)/(f_t) \xrightarrow{\cdot f_t} \mathcal{M}(S, \mathbb{Z}((T))_r)/(f_t^2) \to$$

$$\to \mathcal{M}(S, \mathbb{Z}((T))_r)/(f_t) \to 0.$$

<u>is</u> the extension

$$0 \to \mathcal{M}_p(S) \to \bullet \to \mathcal{M}_p(S) \to 0$$

constructed via entropy.

In fact, "real B_{dR}^+" = "completion at (f_t) of $\mathbb{Z}((T))_r$".

<u>Cor.</u> $\mathcal{M}(S, \mathbb{Z}((T))_{>r})/(f_t) = \mathcal{M}_{<p}(S, \mathbb{R})$.

$\rightsquigarrow$ <u>enough:</u> <u>Theorem</u>.

$$\left(\mathbb{Z}((T))_{>r},\ S \mapsto \mathcal{M}(S, \mathbb{Z}((T))_{>r})\right)$$

defines an "analytic ring".

$\rightsquigarrow$ theory of liquid $\mathbb{Z}((T))_{>r}$-modules.

In better shape, as $\mathbb{Z}((T))_{>r}$ is "locally profinite"

countable union of profinite.

Geometry of Spec $\mathbb{Z}((T))_{>r}$

__Thm__ (Harbater '80s). The ring $\mathbb{Z}((T))_{>r}$ is a principal ideal domain. The max'l ideals are:

1) For any $0 \neq x \in \mathbb{C}$, $|x| \leq r$,

$$\ker\left(\begin{array}{ccc}\mathbb{Z}((T))_{>r} & \longrightarrow & \mathbb{C} \\ T & \longmapsto & x\end{array}\right).$$

This depends only on x up to complex conj..

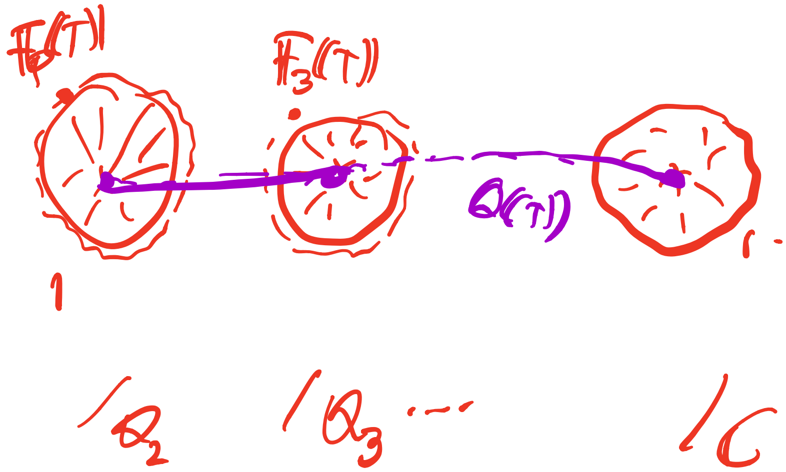
2) For any $0 \neq x \in \overline{\mathbb{Q}}_p$, $|x| < 1$,

$$\ker\left(\begin{array}{ccc}\mathbb{Z}((T))_{>r} & \longrightarrow & \overline{\mathbb{Q}}_p \\ T & \longmapsto & x\end{array}\right).$$

This depends only on $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-orbit of x.

3) For any p,

$(p) = \ker\left(\mathbb{Z}((T))_{>r} \to \mathbb{F}_p((T))\right)$.



<u>Cor.</u> We can also define liquid $\mathbb{Q}_\ell$-vector spaces by base change.

<u>e.g.</u> $\mathbb{Z}((T))_{>r} \longrightarrow \mathbb{Q}_\ell = \mathbb{Z}((T))_{>r}/(T-\ell)$

$T \longmapsto \ell$

Can also describe these liquid $\mathbb{Q}_\ell$-theories in terms of p-measures:

Def'n. Let $\underbrace{0 < p}_{(p \leq 1 \text{ not nec., as } \mathbb{Q}_\ell \text{ nonarch.})} < \infty$. Let for S finite

$$\mathbb{Q}_\ell[S]_{\ell^p \leq c} = \left\{ (x_s)_{s \in S},\ x_s \in \mathbb{Q}_\ell \,\middle/\, \sum |x_s|^p \leq c \right\}.$$

$\ell^0 \leq c$ $\qquad$ $\sup |x_s| \leq c.$

$$\mathcal{M}_p(S, \mathbb{Q}_\ell) = \bigcup_{c > 0} \lim_i \mathbb{Q}_\ell[S_i]_{\ell^p \leq c}$$

$\ell^0 \leq c = \mathbb{Q}_\ell[S]$

$\ell^\infty \leq c = \mathbb{Q}_\ell[S]^\bullet$

$S = \lim_i S_i$

$$\mathcal{M}_{<p} = \bigcup_{p' < p} \mathcal{M}_{p'}.$$

Cor. $(\mathbb{Q}_\ell, \mathcal{M}_{<p})$ is an analytic ring for all $0 < p < \infty$.

all cond. $\mathbb{Q}_\ell$-v.s.

"gaseous". $\qquad$ liquid. $\qquad$ solid.

$p = 0$ $\qquad$ $0 < p < \infty$ $\qquad$ $p = \infty$

Cor $\left(\mathbb{Z}((T^{\mathbb{R}}))_{>r},\ \mathcal{M}(S, \mathbb{Z}((T^{\mathbb{R}}))_{>r}) \right)$ analytic ring.

Prop. $^{\mathbb{R}_{21}}$ The forgetful functors

$$\mathrm{Liq}_p(\mathbb{R}) \longrightarrow \mathrm{Cond}(\mathbb{Z})$$

$$\mathrm{Liq}(\mathbb{Z}((T))_{>r}) \longrightarrow \mathrm{Cond}(\mathbb{Z}[T^{-1}])$$

are fully faithful.

equiv.: $$\left(\mathbb{Z},\ S \mapsto \mathcal{M}_{<p}(S,\mathbb{R})\right)$$

$$\left(\mathbb{Z}[T^{-1}],\ S \mapsto \mathcal{M}(S, \mathbb{Z}((T))_{>r})\right)$$

define analytic rings.

Sketch of proof: $\left(\mathbb{Z}[T^{-1}],\ S \mapsto \mathcal{M}(S, \mathbb{Z}((T))_{>r})\right)$

is an analytic ring.

Prop'n. $\mathcal{M}(S, \mathbb{Z}((T))_r)$ is a pseudocoherent $\mathbb{Z}[T^{-1}]$-module, i.e. admits infinite resolution with all terms compact projective.

Proof. Let $\underset{\circlearrowleft T^{-1}}{M} = \mathcal{M}(S, \mathbb{Z}((T))_r)$.

Breen-Deligne resolution

$$\cdots \to \underset{\circlearrowleft T^{-1}}{\mathbb{Z}[M^2]} \to \underset{\circlearrowleft T^{-1}}{\mathbb{Z}[M]} \to \underset{\circlearrowleft T^{-1}}{M} \to 0$$

resolution as cond. $\mathbb{Z}[T^{-1}]$-modules.

enough: $\mathbb{Z}[M^r]$, or really $\mathbb{Z}[M]$ pseudocoherent $/\mathbb{Z}[T^{-1}]$.

Have profin. subset $M_{\le 1} \subseteq M$,

$$M = \bigcup_{n \geq 0} T^{-n} M_{\le 1}. \qquad M_{\le 1} = T \cdot M_{\le 1}.$$

$$0 \to \mathbb{Z}[T^{-1}][M_{\le 1}] \xrightarrow{T^{-1} - [T^{-1}]} \mathbb{Z}[T^{-1}][M_{\le 1}] \to \mathbb{Z}[M] \to 0$$

$\mathbb{Z}[T^{-1}]$-lin. ext. of $\mathbb{Z}[M_{\le 1}] \subseteq \mathbb{Z}[M]$

free on profin., thus pseudocoh.

$\square$

What do we have to prove?

(*) (kernel version) If K kernel of some map

$$\bigoplus_i \mathcal{M}(S_i, \mathbb{Z}((T))_{>r}) \to \bigoplus_j \mathcal{M}(T_j, \mathbb{Z}((T))_{>r}),$$

then

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}[T^{-1}]}(\mathcal{M}(S, \mathbb{Z}((T))_{>r}), K) \xrightarrow{\sim} R\underline{\mathrm{Hom}}_{\mathbb{Z}[T^{-1}]}(\mathbb{Z}[T^{-1}][S], K).$$

Slightly better: $\forall\ 1 > r' > r > 0$

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}[T^{-1}]}(\mathcal{M}(S, \mathbb{Z}((T))_{r'}), K) \xrightarrow{\sim} R\underline{\mathrm{Hom}}_{\mathbb{Z}[T^{-1}]}(\mathbb{Z}[T^{-1}][S], K).$$

Prop'n $\Rightarrow$ Both sides commute with filtered colimits in K.

$\rightsquigarrow$ reduce to

$$K = \ker\left(\mathcal{M}(S_1, \mathbb{Z}((T))_r) \to \mathcal{M}(S_2, \mathbb{Z}((T))_r)\right).$$

"r-Smith space".

further trick (uses variations in r, product trick).

reduce to $K = V =$ "r-Banach".

<u>Def'n.</u> An r-Banach $\mathbb{Z}[T^{-1}]$-module V is a top. $\mathbb{Z}[T^{\pm 1}]$-module V s.th. $\exists$ norm $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$

i) $\|v\| = 0 \iff v = 0$

ii) $\|v+w\| \leq \|v\| + \|w\|$.

iii) $\|T^{-1} v\| = r^{-1} \|v\|$.

iv) complete

v) $|\cdot|$ defines topology.

Any such V is automatically $\mathbb{Z}((T))_{>r}$-module.

<u>Key Result</u>. For r-Banach V, $1>r'>r>0$

$$\begin{array}{c} R\underline{\mathrm{Hom}}_{\mathbb{Z}[T^{-1}]}\left(\mathcal{M}(S,\mathbb{Z}((T))_{r'}),V\right) \\ \downarrow{\scriptstyle 2} \\ R\underline{\mathrm{Hom}}_{\mathbb{Z}[T^{-1}]}\left(\mathbb{Z}[T^{-1}][S],V\right). \end{array}$$

<u>rough idea:</u> Any r'-measure μ on S, $\leq c$, can be written as a sum $\sim N$ measures μ_i each of which is $\lesssim \frac{c}{N}$.

+ finitely many "big" terms.

different scaling behaviour of T on source $\mathcal{M}(S,\mathbb{Z}((T))_{r'})$ and target V

$\rightsquigarrow$ μ_i will act by something very small.
action of finitely many big terms
is already given by map $\mathbb{Z}[T^{-1}][S] \to V$.

<u>Remark</u>. Using Breen - Deligne resolution,
it is "impossible" to prove
$$A = B$$
through only computing A via BD resolution.
(as result inexplicit).
Instead, need to rewrite in form
$$C = 0,$$
compute C via BD resolution.

$\rightsquigarrow$ Let $\overline{\mathcal{M}}_{r'}(S) = \mathcal{M}(S, \mathbb{Z}((T))_{r'}) / \mathbb{Z}[T^{-1}][S]$.

<u>Want</u>: $\underline{\mathrm{RHom}}_{\mathbb{Z}[T^{-1}]}(\overline{\mathcal{M}}_{r'}(S), V) = 0.$

Good News. $\overline{\mathcal{M}}_{r'}(S)$ still a quasi-sep cond. set.

$$\overline{\mathcal{M}}_{r'}(S) = \bigcup_{c>0} \lim_i \underbrace{\big(T\mathbb{Z}[[T]]_{r'}[S]\big)_{\le c}}_{\overline{\mathcal{M}}_{r'}(S)_{\le c}} .$$

$\subseteq$ closed $\mathbb{Z}((T))_{r'}[S]_{\le c}$

Use:

$$\mathbb{Z}[S] = \bigcup_{c>0} \lim_i \mathbb{Z}[S_i]_{\ell \le c}$$

Now use Breen-Deligne resolution.

$$\cdots \to \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)^2] \to \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)] \to \overline{\mathcal{M}}_{r'}(S) \to 0$$

resolution as $\mathbb{Z}[T^{-1}]$-module.

$$0 \to \mathbb{Z}[T^{-1}][\overline{\mathcal{M}}_{r'}(S)_{\le c'}] \xrightarrow{T^{-1}-[T^{-1}]} \mathbb{Z}[T^{-1}][\overline{\mathcal{M}}_{r'}(S)_{\le c}] \to \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)] \to 0$$

Problem: not exact.

For certain $c_1, c_2, \dots, c_1', c_2', \dots$

$$(*)_{c_i, c_i'}:\quad \begin{tikzcd} \cdots \arrow[r] & \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)^2_{\leq c_2'}][T^{-1}] \arrow[r] \arrow[d] & \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)_{\leq c_1'}][T^{-1}] \arrow[d] \\ \cdots \arrow[r] & \mathbb{Z}[\widehat{\mathcal{M}}_{r'}(S)^2_{\leq c_2}][T^{-1}] \arrow[r] & \mathbb{Z}[\widehat{\mathcal{M}}_{r'}(S)_{\leq c_1}][T^{-1}] \end{tikzcd}$$

colimit of all these gives resolution of $\overline{\mathcal{M}}_{r'}(S)$.

For each choice of c_i, c_i', can compute

$R\mathrm{Hom}((*)_{c_i,c_i'}, V)$: will not be acyclic,

but transition maps ought to kill cohomology.

"Precise" statement one proves by induction:

Given cohom. degree, can increase all c_i, c_i' to

cc_i, cc_i' (c depends only on cohom degree)
s.th. the transition map is 0 on
$\underline{\mathrm{Ext}}^i$; more precisely, can even bound
size of the preimage under differential.

How does one get preimages under
differential?

$$\overline{\mathcal{M}}_{r'}(S)^N_{\le c/N+\epsilon} \xrightarrow{\ \Sigma\ } \overline{\mathcal{M}}_{r'}(S)_{\le c}$$

essentially surj.

$\rightsquigarrow$ build a double (triple) complex

$$\begin{array}{ccccc}
(*)_{c_i, c_i'} \to & \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)^2_{\le c'}] & \longrightarrow & \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)_{\le c}] & [a_1+\dots+a_N] \\
 & \uparrow \Sigma & & \uparrow \Sigma & \uparrow \\
\dots \longrightarrow & \mathbb{Z}[(\overline{\mathcal{M}}_{r'}(S)^2_{\le [illegible]})^N] & \longrightarrow & \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)^N_{\le c/N}] & [(a_1, \dots, a_N)]
\end{array}$$

(similar complexes)

"enough by induction": $\Sigma \overset{?}{=} 0$ on Ext^i

(or at least very small).

<u>Key</u>. Σ homotopic to Σ on the outside.

Summing on outside leads to

very small map

(by different scaling behaviour of T or V).