

# Condensed K-theory cont'd

Recall Suslin's theorem:

Thm:  $K(\mathbb{C}^{\delta}) \rightarrow KU$

~~map~~  $\nearrow$  mysterious

gp completion  
of category  
enriched in  
animo  
 $(BGL_n(\mathbb{C}))$

$\pi_n KU = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

is an iso with  
finite coeffs:

$$K(\mathbb{C}^{\delta})/n \xrightarrow{\sim} KU/n$$

$$(X \xrightarrow{n} X \rightarrow X/n)$$

---

Now  $K(\mathbb{C})$ : condensed  
spec<sub>u</sub>.

s.t: 1)  $K(\mathbb{C})/n = K(\mathbb{C}^{\delta})$

$$2) \kappa(\mathbb{C})^* = \kappa_U$$

$$(\text{or } h(\kappa(\mathbb{C})) = \kappa_U)$$

Refined Justification:

$$\kappa(\mathbb{C})/n \quad \mathbb{B}$$

discrete.

$$\Rightarrow \kappa(\mathbb{C})/n \stackrel{\approx}{\sim} \kappa(\mathbb{C})/n$$

|| discrete

$$(\kappa(\mathbb{C})/n)^*$$

||

$$\kappa_U/n \approx \kappa(\mathbb{C})^*/n$$

Lemma: A condensed <sup>anims</sup> spectrum

$X$  is discrete

$\Leftrightarrow \forall$  extr. disc.  $S,$

$s \in S$ ,

$$\lim_{U \ni s} X(U) \rightarrow X(\{s\})$$

$X(\{s\})$

$\Rightarrow$  an iso.

Def:  $A$ : ordinary spectrum,  
 $\Leftarrow$  soc. discrete and  
spectrum

$A^S(S) =$  ~~sheaf~~  
global sections  
of const shf  
 $A$  on top  
space  $S$ .

$\Rightarrow$  can be an  $\mathbb{A}^1$  discrete  
 $\Leftrightarrow U \mapsto X(U)$   
 $\mathbb{A}^1$  const shf on  
open  $U \subset S$ .

$$X(\{s\}) \rightarrow X$$

can check  $\mathbb{A}^1$  by  
- 1 - 11 -

Stalks.

Take  $X = \mathbb{A}^1_{\mathbb{C}}/n$

$\lim_{\substack{\longrightarrow \\ U \ni s}} k(\text{cont}(U, \mathbb{C}))/n$

$\uparrow$   
closed  
open

$\cong k(\text{germs of cts  
functions  
(S, s) \rightarrow \mathbb{C}})/n$

Note: Ring of germs  
of cont's functions  
 $\Rightarrow$  a Henselian local  
ring w/ residue  
field  $\mathbb{C}$ ,  
via  $eU_s$ .  
(implicit inverse function)

Swan's rigidity thm for  
Henselian local rings

$\Rightarrow k[R] \rightarrow k[R/\mathfrak{m}]$

$\frac{1}{n} \dots \frac{1}{n} \dots \frac{1}{n}$

---

No  $\forall$  such  $K$ ,  
maximized  $\sigma$ .

---

Probably  $A/\mathbb{Q}$

$$\Rightarrow K(A)^{\circ} = K^{\text{st}}(A)$$

$$\stackrel{ii}{=} h(K(A))$$

---

Same argument  $\Rightarrow$

$$K(A)/\mathbb{Q} \quad \text{is}$$

dense for  
any Banach ring  $A$   
(in char 0)  
with  $x \in A$

e.g.  $A = \mathbb{Q}_p$ .

$$K(\mathbb{Q}_p)^\bullet = ?$$

---

Recall: localization sequence

$$K(\mathbb{R}_p) \rightarrow K(\mathbb{Q}_p) \rightarrow \sum K(\mathbb{F}_p)$$

of condensed spectra.

↑  
discrete

$$K(\mathbb{Q}_p)^\bullet = \varprojlim_n K(\mathbb{Q}_p/n)$$

↑  
discrete.

---

Background:

$$K(\mathbb{C}), K(\mathbb{Q}_p),$$

$$K(\mathbb{Q}_p)$$

these are  $\text{Sng} \mathcal{L}$

reasonable in  
the same way as

$$\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathbb{Q}_p.$$

E.g.  $\therefore K_2(F) = \frac{F^{\times} \otimes F^{\times}}{\mathbb{Z}}$

But they all way to more  
targets, e.g.

if you take these  
coeffs, or

~~K~~ regulator maps

$$K_{\text{ant},1}(\mathbb{R}) \rightarrow \mathbb{R}.$$

You'd like to know  
what the initial  
reasonable thing is  
that  $K(\mathbb{R})$   
maps to.

(Related: alg cycles)

↓ -

Weil Coh  
theory

⇒ Good guess for

$K(\mathbb{Q})$  reasonable

$n = 0, 1, 2, 3$

$\mathbb{Z} \quad \mathbb{C}^x \quad 0 \quad \mathbb{C} / (\mathbb{Z} + i)\mathbb{Z}^0 =$

$\mathbb{C} / (\mathbb{Z} + i)\mathbb{Z} = \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}, \mathbb{Z})$

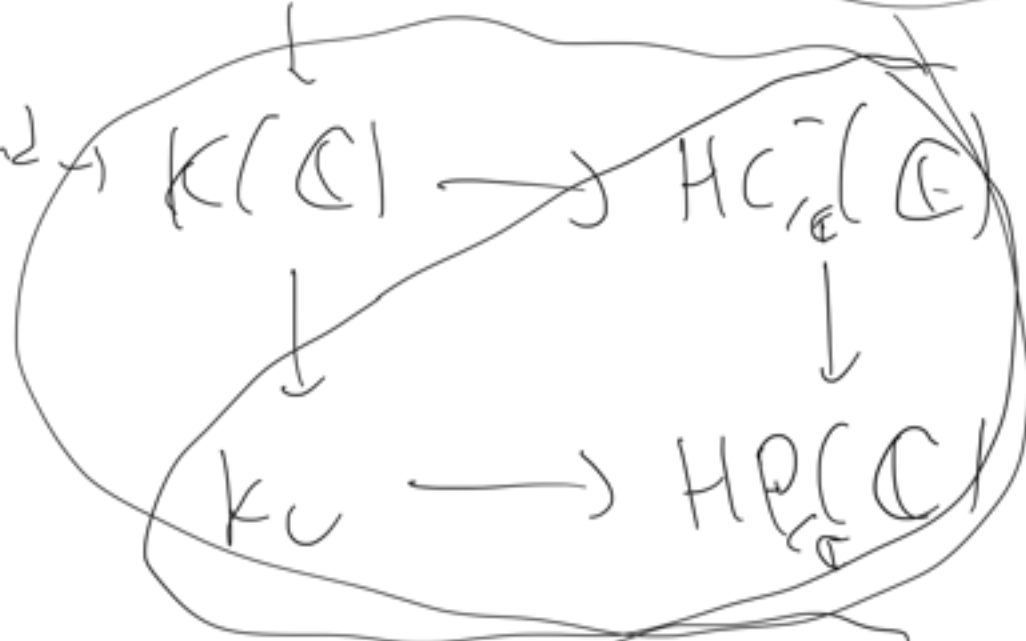
How to recover the  
archimedean  
information?  
I.e. the regulator?



# Open problem.

$k \langle \sigma \rangle$

closed



$k(C) \longrightarrow$



## Morava E-theory:

$k$ : perfect field of  
char  $p$

$G$ :  $\mathbb{Z}$ -linear  
let  $n$  formal  
gp /  $k$

$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

$$f(x, y, z) = f(x, f(y, z), \dots)$$

More E-thy Goerss-Mopkins-  
Miller,  
...

$$E(k, G)$$

||

$$E_n$$

$E_n$  is a  $E_\infty$   
ring spectrum

(analog of commutative  
ring in spectra)

of a very special kind:

It's even ( $\pi_{odd} E_n = 0$ )

& periodic ( $\exists v \in \pi_2 E_n$   
s.t.

$$\pi_x E = (\pi_0 E) [E^{(x)}]$$

With

$$\begin{array}{ccc} & A \rightarrow k & \\ & \uparrow \quad \uparrow & \\ \pi_0 E & \cong W(k)[x_1, \dots, x_m] & \end{array}$$

= complete local  
ring parameterized  
deformations  
of  $G$   
complete.

Carries universal  
deformed formal  
gp  $G_{\text{univ}}$ .

no longer of ht exactly  
 $n$ , rather ht  $\leq n$ .

OTOH, any even  
periodic ring spectrum  
is a commutative

(with ...)

formal gp on  $\pi_0$ .

[Quillen]

& for Morava E-theory

these agree.

On the side of spectra,  
this is a series of  
approximations  $\hookrightarrow$

$$S \rightarrow \dots \rightarrow L_n S \rightarrow L_{n+1} S$$

$$\dots \rightarrow L_0 S$$

$$\parallel \\ H\mathbb{Q}$$

Hopkins-Ravenel

$$\boxed{S_{(p)}} \xrightarrow{\sim} \varprojlim_n \boxed{L_n S}$$

$$\boxed{L_n S} \in \langle E_n, E_n \otimes E_n, \dots \rangle$$

$$\left[ \dots \right]^{(p)}$$

$$E_a \otimes E_{a+1} \dots$$

$$E_n \otimes \dots \otimes E_n$$

(Hopkins-Rannell)

Idea: the ~~stabilization~~ filtration of formal

groups by height

is reflected strongly

in the inverse structure of

$S_p$

(Ravenel, <sup>Devriatz</sup> Hopkins-Smith)

$$\pi_0(E_n) = \boxed{W(k) \langle v_1, \dots, v_n \rangle}$$

↑  
calculated

Top version:

Q:

Q: How to produce a condensed  $E_{\infty}$ -ny spectrum  $E_n$

s.t.  $\pi_0 E_n = \text{wcl}(E_{\infty})$

holds as  
Cond. ab gps?

Top version: How to produce  $\tau_p$  as a condensed ny?

2 answers:

1)  $\tau_p$  - lim  $\tau_{p,n}$

$$1) \mathcal{O}_P = \varprojlim_n \mathcal{O}_P^{(n)}$$

$\uparrow$                        $\uparrow$   
 in comb rings,              discrete

$$= \left( \mathcal{O}_P^\delta \right) \wedge_P$$

$\uparrow$   
 in the condensed world.

$$2) S \mapsto W(C(S, \mathbb{F}_p))$$

$\uparrow$   
 perfect  $\mathbb{F}_p$ -algebra

unimodular flat  $\mathbb{Q}_p$  algebra  
 (lifting  $C(S, \mathbb{F}_p)$ ).

$$C(S, \mathbb{Q}_p) = \varprojlim_n C(S, \mathbb{Q}_p^{(n)})$$

---

Likewise for  $\mathbb{F}_p$ .

It turns out that  
 for spectra,  
 there's more than  
 just  $\mathbb{P}$  for  
 (an intrinsically)  
 complete st.

There's a whole  
 sequence

$\mathbb{P}, \mathbb{V}_1, \mathbb{V}_2, \dots$

$\mathbb{P} \leftarrow \mathbb{V}_0 \leftarrow \mathbb{V}_1 \leftarrow \mathbb{V}_2 \leftarrow \dots$

$\uparrow$   
 $\in [\Sigma^d \mathbb{S}/\mathbb{P}, \mathbb{S}/\mathbb{P}]$

acts in a degree-  
 shifting way

$\rightarrow \mathbb{X}/\mathbb{P}$



for any  $X \in Sp$

$$v_2 : \sum^d x_i e_i v_i \rightarrow x_i e_i v_i$$

... ..

→ you can complete  
any spectrum at  
( $p_i v_i \rightarrow v_{n-c}$ ).

(formally, Bousfield  
loc.)  
with  $S/p_i v_i \dots v_{n-c}$

up to multiply  
by a power of  $v$ ,

~~map~~  $p_i v_0, v_1, \dots, v_{n-1}$   
 $\in \pi_0 E_n$

~~the~~ ~~cost~~  
 have ~~(1)~~ ~~to~~ generate  
 an open ideal

$\Rightarrow$  copleting  $\hat{E}$   
 from  $\hat{E}$  is the  
 same as copleting  
 $\hat{E}$  at  $(1)$ .

Option 1:

$E_n^\delta$

$(E_n^\delta)_{\wedge_{P_1 \dots P_n}}$   
 $\uparrow$   
 in the completed  
 world.

or  $(E_n^\delta)_{\wedge_{P_1 \dots P_n}}$

don't worry in

$E_n^\delta$  - modules in

Cond SP.

Option 2:

Loric:  $(k, G) \rightsquigarrow E(k, G)$

works also for  $\pi_V^-$   
 $w(k) \{v_1, \dots, v_n\}$

$k$  a perfect ring

&  $G$  a formal gp of  
 $ht = n$

$$\boxed{E_{gr}(S)} = E(c(S, k), \overset{\text{pullback}}{\text{of}} G)$$

$\uparrow$   
perfect ring

$$E_n^{\delta} \rightarrow E_n$$

both are  $(P, v_1, \dots, v_n)$ -  
complete,

$S, c \geq n$  check

$\Rightarrow$  both are discrete,  
 hence too.

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$$L_n \subseteq \langle E_n, E_n \otimes E_{n-1} \rangle$$

$$\downarrow \wedge (p, v_1, \dots, v_n)$$

$$L_k(n) \subseteq \langle E_n \rangle$$

Can be more  
 precise about  
 how to build

$L_k(n) \subseteq$  from  $E_n$ .

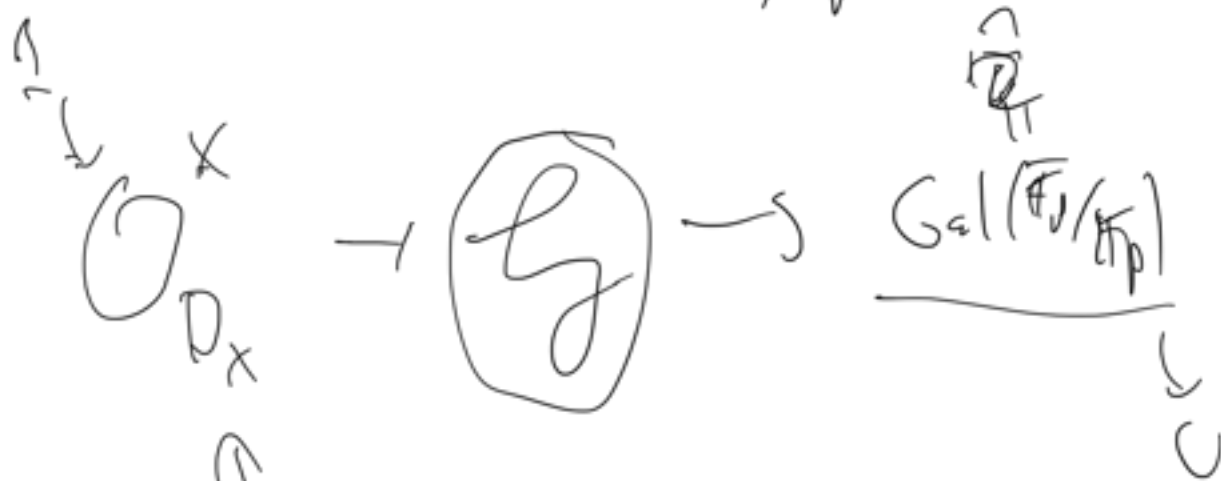
$$k = \overline{\mathbb{F}_p}, \quad G = \text{finite group of order } n.$$

$\exists$  a non-sol... / +

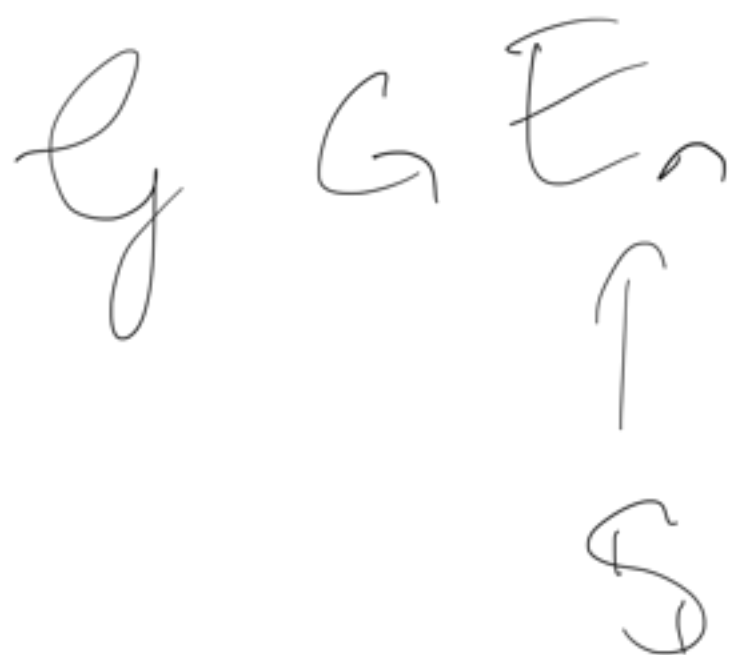
→ group scheme / Hopf  
 paravectorial automorphisms  
 product of pair  $(k, G)$   
 $G \rightarrow k$   
 $k \rightarrow k$

$$= \underline{G}$$

$G$ : (Extended)  
 Maximal  
 stabilizer  
 of  $\mathcal{P}$



units in ring of integers  
 of division alg  
 of  $m$ . in



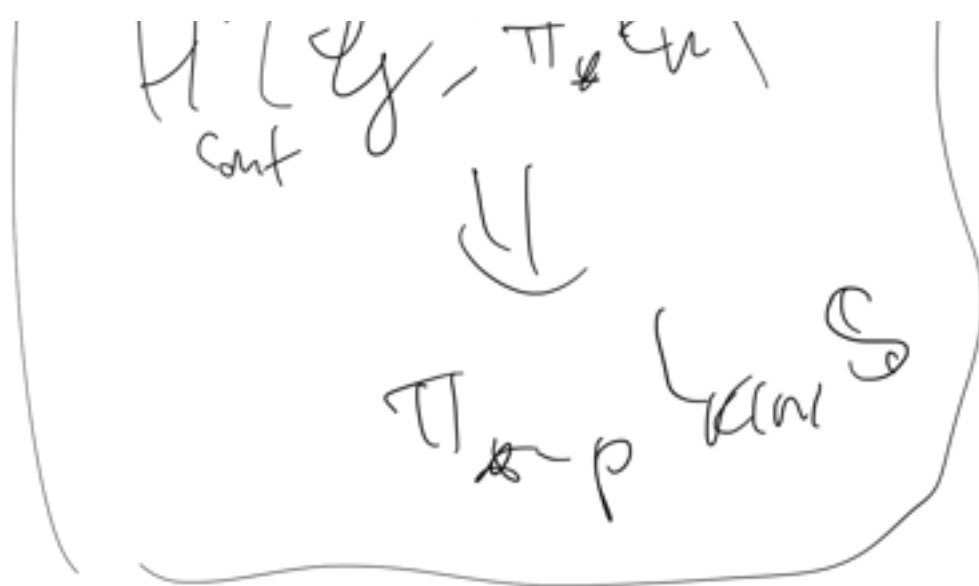
$$L_{K(n)} \xrightarrow{S} E_n \text{ "hcy"}$$

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Problem: for this to be true, it needs to be continuous w/pt fixed pts on right.

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1. P/A . . .



$l y G \pi \& E$   
 is not centered  
 w/out dimer sep  
 or  $\pi \& E$   
 & prof sep w/out.

what ~~the~~ does,  
 it mean that  
 $l y$  acts as  $E_n$   
 $\uparrow$  condensed  
 gp  $\uparrow$   
 condensed  
 spectrum



$G \hookrightarrow A$   
cond as  
gp

$G \times A \rightarrow A$   
cond sets

~~then~~ s.t.  
an S-valued  
pts,

get

$G(S) \times A(S) \rightarrow A(S)$   
group action on  
an ab gp.

Could say same thing:

$G \rightarrow \underline{\text{Act}}(E_n)$   
condensed  
gp action

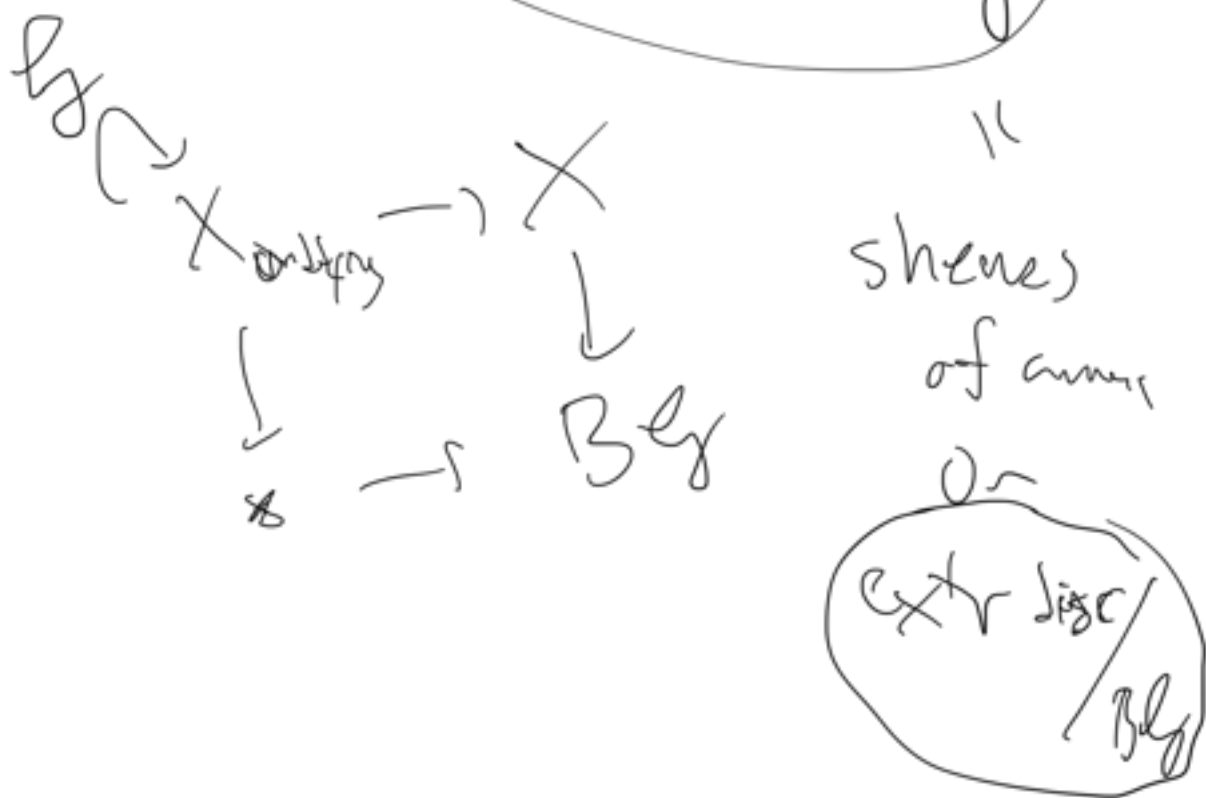


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Different perspective:

use slice topos

$\text{CondAn} / \text{Bgl}$



$S, \mathcal{G}$ -torsor on  $S$ .

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$S$  extr disc patch set,

$X$   
 $\downarrow \mathcal{G}$ -torsor  
 $S$

$X$   
 $\downarrow \mathcal{G}$ -torsor

$$\Rightarrow \downarrow \overset{\sim}{\cong}$$

$$\underline{S} = \text{const. gp scheme} \\ = \text{Spec}(C(S, k))$$



$$\underline{S} \rightarrow \mathcal{B}G$$

" stack of formal gps  
 No to  $G_{\text{pro}}$

$\Rightarrow$  classifies a mu formal gp in  $\underline{S}$

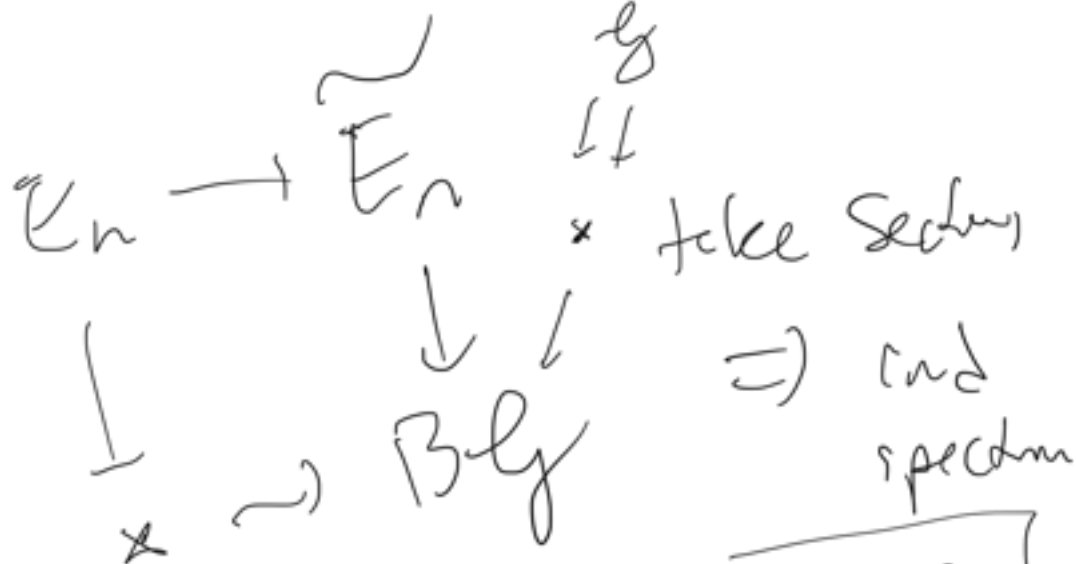
$$\Rightarrow E(C(S, k), G')$$

How to check

$$k(S) \xrightarrow{\sim} E_n^{hy}$$

T  
condensed spectrum.

$$L_n S \sim P_{1 \sim n} V_{n+1}$$



$$\boxed{E_n^{hly}}$$

Write  $E_n^{hly}$  ( )

K(n)-local:

$$E_n^{hly} = \varprojlim (E_n \rightarrow C(\mathbb{Q}, E_n))$$

$\downarrow$   
 $C(\mathbb{Q}, E_n)$   
 $\downarrow$   
 $\downarrow$

enough to check  
mod  $p_1, \dots, p_m$

$$L_n S / p_1, \dots, p_m \xrightarrow{hg} E_n / p_1, \dots, p_m$$

can ( $\otimes E_n$ )  
dede n



$$C(G, E_n / p_1, \dots, p_m) \xrightarrow{hg} E_n / p_1, \dots, p_m$$

cl

$$E_n / p_1, \dots, p_m$$

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Rk:



$E_n$  il

sol: 1

$$E_1 = KU_n$$

$$E_1 \otimes E_1 \text{ not } p\text{-complete!}$$

( $E_1$  non-connected)

$$L_n \mathcal{S} \quad E_n \quad E_n \otimes E_n$$

Rk: We just talked about condensed spectra w/

$\mathbb{I}$ -action,

$\mathbb{I} = \text{profinite}$   
 $\mathbb{I} = \mathbb{Z}_p$

h. d. n' h. d. i

what is the theory  
w/ previous  
approach?

$$(E_n)^{\mathbb{Z}} \xrightarrow{I} \text{Locus } \mathcal{S} = \text{" } E_n^{\text{h}\mathbb{Z}} \text{"}$$

Devil's-tail-Hopkins: ~~max~~



Produced a <sup>hyper</sup>sheaf  
of  $E_\infty$ -ring spectra  
on the site of  
finite continuous

$\mathbb{T}$ -sets

S.t. :

1) on  $\mathcal{S}$ , if

$\text{Locus } \mathcal{S}$

2) Have descent SS

1. D. ...

$H'_{\text{cont}}(I; \mathbb{R}) \cong H'_{\text{cont}}(I; \mathbb{R})$

3) stalk @ pt  
of this site

$$= \varinjlim_{U \ni I} "E_n"$$

is something whose  
localization

~~is~~  $E_n$ .

---

then  $X$  discrete  
anima.

2 ways of saying

$\mathbb{I} \hookrightarrow \mathbb{C} \rightarrow X$ :

1) view  $X$  as condensed,  
take what we  
discussed

2) Find a sheaf  
 on finite cpts  
 $\mathcal{I}$  = sets w/  
 stalk  $X$ .

Claim:  $\infty$ -cut of  
 $n$ -trunct discrete cond prime  
 w/ action of  $\mathcal{I}$

$\approx$  Postnikov completion

$n$ -trunct  
 of  $\infty$ -cut  
 of sheaves of  
 anima on  
 site of finite  
 $\mathcal{I}$ -sets

$\approx \lim_{\leftarrow} (Sh_{\mathcal{C}_n}(X) \dashv Sh_{\mathcal{C}_n}(\mathcal{I}) \dashv Sh_{\mathcal{C}_n}(X))$

or  $\left( \right)$



shears of  
anime on  
top space  $\mathbb{I}$ .

Rk: if  $\text{vcd}(\mathbb{I}) \leq \infty$

(e.g.  $\mathbb{I} = \mathbb{R}^1$ ,

then Postnikov  
completion

= hypersheaf).

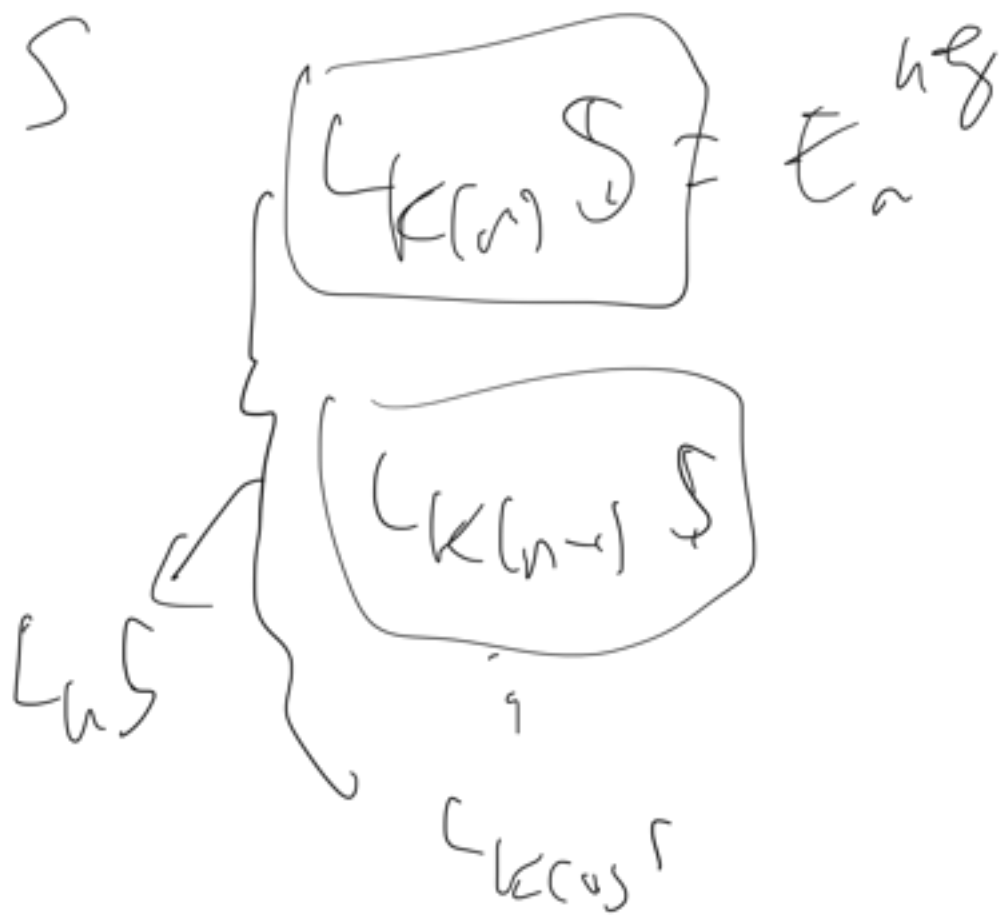
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Rk: This is an  
setting where  
the condensed  
approach is the

only "reasonable"  
one:

transchromatic

http theory.



$$L_{K(n-2)} L_{K(n-1)} E_n$$

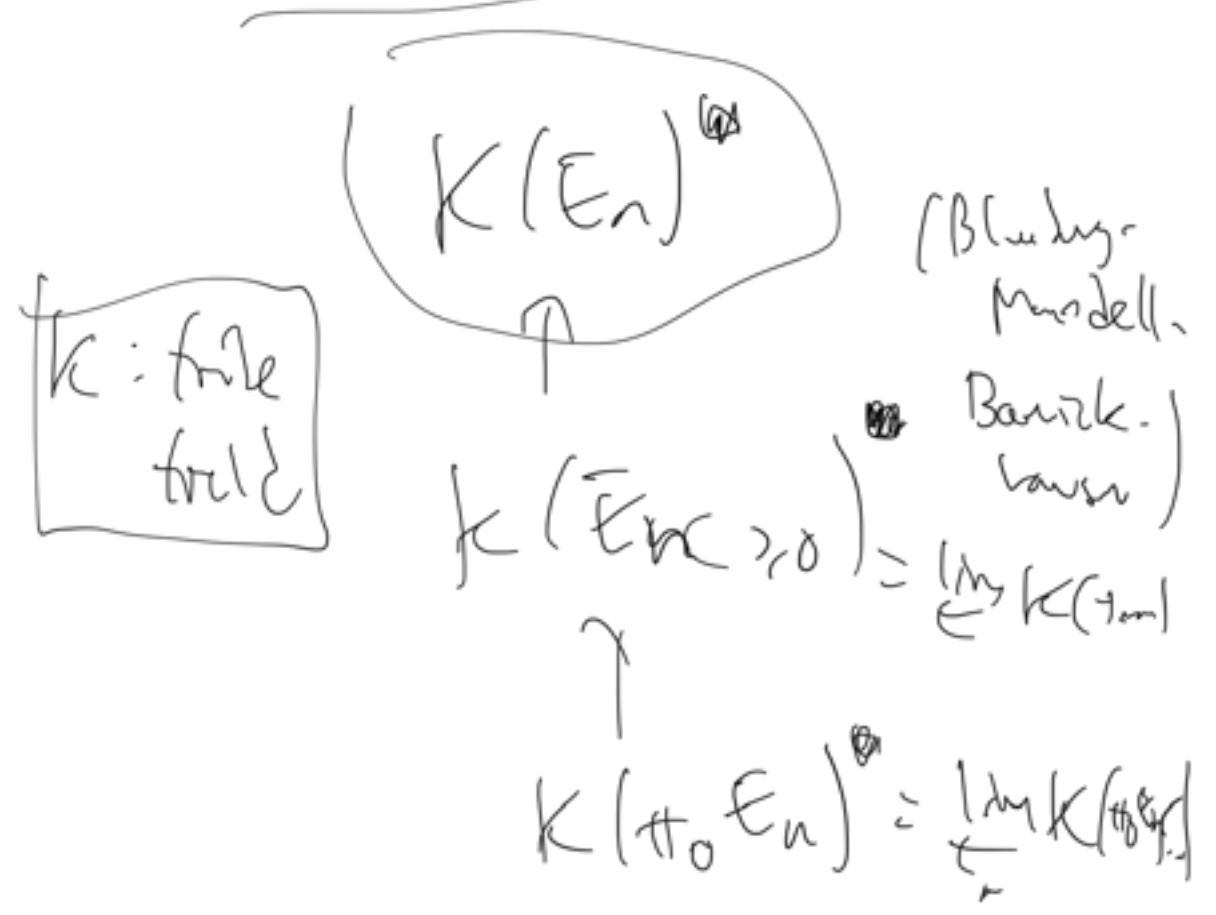
$$\Pi_0 L_{K(n-1)} E_n$$

$$\approx \underbrace{w(K) [E_n, \dots, E_n]}_{\text{dim } (w(K) \Pi_0 \dots)}$$

$$\text{dim } (w(K) \Pi_0 \dots)$$



is not a top ring,  
 complete ness  
 to study  
 classically.



~~AB~~

