Condensed K-theory cont'd

Recall Suslin's theorem:

Thm: $K(\mathbb{C}^\delta) \to ku$

mysterious

ku $=$ gp cpletion of category enriched in anima $|BGL_n(\mathbb{C})|$

$\pi_n ku = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

is an iso with finite coeffs:

$$K(\mathbb{C}^\delta)/n \xrightarrow{\sim} ku/n$$

$$\left(X \xrightarrow{n} X \to X/n \right)$$

Now $K(\mathbb{C})$: condensed spectrum.

s.t.: 1) $K(\mathbb{C})(*) = K(\mathbb{C}^\delta)$

$$2)\quad K(\mathbb{C})^{\blacksquare} = ku$$

$$\left(\text{or } h(K(\mathbb{C})) = ku\right)$$

Refined Suslin thm:

$K(\mathbb{C})/n$ is discrete.

$$\Rightarrow\quad K(\mathbb{C})(*)/n \xrightarrow{\sim} K(\mathbb{C})/n$$

$$\| \text{ discrete}$$

$$\left(K(\mathbb{C})/n\right)^{\blacksquare}$$

$$\|$$

$$ku/n = K(\mathbb{C})^{\blacksquare}/n$$

Lemma: A condensed ~~spectrum~~ anima X is discrete

$\Leftrightarrow$ $\forall$ extr. disc. S,

$s \in S$.

$$\varinjlim_{U \ni s} X(U) \to X(\{s\}) = X(*)$$

is an iso.

<u>Pf:</u> A: ordinary spectrum, assoc. discrete cond spectrum

$A^\delta(S) =$ ~~shf~~ global sections of const shf A on top space S.

$\Rightarrow$ Cond an is discrete $\Leftrightarrow$ $U \mapsto X(U)$ is const shf on opens $U \subset S$.

$$X(*) \dashv X$$

can check this by

Stalks.

Take $X = K(\mathbb{C})/n$

$$\varinjlim_{\substack{U \ni s \\ \text{closed} \\ \text{open}}} K(\mathrm{Cont}(U, \mathbb{C}))/n$$

$$= K\left(\begin{array}{c}\text{germs of cts} \\ \text{functions} \\ (S, s) \to \mathbb{C}\end{array}\right)/n$$

<u>Note:</u> Ring of germs of cont's functions is a <u>henselian local ring</u> w/ residue field $\mathbb{C}$, via ev_s.
(implicit inverse fn thm)

Suslin's rigidity thm for henselian local rings

$$\boxed{\Rightarrow \quad K(R) \to K(R/\mathfrak{m})}$$

[illegible] $/n$ [illegible] $/n$

No $\forall$ such R.

may need $\overline{cR}$.

Probably $A/\mathbb{C}$

$$\Rightarrow K(A)^{\circ} = K^{st}(A)$$
$$\| $$
$$h(K(A))$$

Same argument $\Rightarrow$

$K(A)/n$ is

discrete for

any Banach ring A

(in char 0,

with $n \in A$)

e.g. $A = \mathbb{Q}_p$.

$$K(\mathbb{Q}_p)^{■} = ?$$

Recall: localization sequence

$$K(\mathbb{Z}_p)^{■} \to K(\mathbb{Q}_p)^{■} \to \Sigma K(\mathbb{F}_p)$$

of condensed spectra.

discrete

$$K(\mathbb{Z}_p)^{■} = \varprojlim_n K(\mathbb{Z}/p^n)$$

discrete.

Background:

$$K(\mathbb{C}), \quad K(\mathbb{C}_p),$$

$$K(\mathbb{Q}_p)$$

these are big &

unreasonable in the same way as $\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

E.g.: $K_2(F) = F^\times \underset{\mathbb{Z}}{\otimes} F^\times / \text{rel'n}$

But they all map to more targets, e.g. if you take finite coeffs, or

regulator maps

$K_{4n+1}(\mathbb{R}) \to \mathbb{R}$.

You'd like to know what the initial reasonable thing is that $K(\mathbb{R})$ maps to.

(Related: alg cycles

$\downarrow$ –

Weil Coh theory)

$\exists$ Good guess G–

$\boxed{K(\mathbb{C})^{\text{reasonable}}}$:

$n = 0, 1, 2, 3$

π_n $\quad \mathbb{Z} \quad \mathbb{C}^\times \quad 0 \quad \boxed{\mathbb{C}}/(2\pi i)^2\mathbb{Q}^0$:

$$\boxed{\mathbb{C}}/2\pi i\mathbb{Q} = \mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), \mathbb{Q}(1))$$

How to recover the archimedean information?

I.e. the regulator?

Open problem.

$$\begin{tikzcd} K(\mathbb{C})^{S} \arrow[r] \arrow[d] & 1 \\ K(\mathbb{C}) \arrow[r] \arrow[d] & HC^{-}_{/\mathbb{C}}(\mathbb{C}) \arrow[d] \\ ku \arrow[r] & HP_{/\mathbb{C}}(\mathbb{C}) \end{tikzcd}$$

cartesian $\to$

$$K(\mathbb{C}) \longrightarrow HC^{-}(\mathbb{C}) \times_{HP(\mathbb{C})} ku$$

Morava E-theory:

k : perfect field of char p

G : ht n formal 1-dim'l gp / k

$G(x, y) \in k[[x, y]]$

$f(x) \dots$

$f(f(x,y),z) = f(x,f(y,z))$,

$\dots$

Morava E-theory — Goerss-Hopkins-Miller, ...

$$E(k, G)$$
$$\|$$
$$E_n$$

E_n is a $\underline{E_\infty\text{-}}$ $\underline{\text{ring spectrum}}$

(analog of commutative ring in spectra)

of a very special kind:

It's $\underline{\text{even}}$ $(\pi_{odd} E_n = 0)$

& $\underline{\text{periodic}}$ $(\exists\, v \in \pi_2 E_n$ s.t.

$$\pi_* E = (\pi_0 E)[u^{\pm 1}].$$

With

$$\begin{array}{c} A \longrightarrow k \\ \nwarrow \quad \nearrow \\ \pi_0 E \cong W(k)[[x_1, \ldots, x_{n-1}]] \end{array}$$

= complete local ring parametrizing $\underbrace{\text{deformations}}$ of G $\cancel{\text{complete}}$.

Carries universal deformed formal gp G_{univ}.

no longer of ht $\underline{\text{exactly}}$ n, rather ht $\leq n$.

OTOH, $\underline{\text{any}}$ even periodic ring spectrum carries a canonical formal gp on π_0.

(Quillen)

& for Morava E-thy

these agree.

On the side of spectra,

there is a series of

approximations to

$$S \to \cdots L_n S \to L_{n-1} S$$

$$\cdots \to L_0 S$$

$$= HQ$$

Hopkins-Ravenel

$$S_{(p)} \xrightarrow{\sim} \varprojlim_n L_n S$$

$L_n S \in \langle E_n, E_n \otimes E_n,$

$E_a \otimes E_a, \ldots$

$E_n \otimes \ldots \otimes E_n \rangle$

(Hopkins-Ravenell

Idea: the ~~stack~~ filtration of formal groups by height is reflected strongly in the innate structure of Sp

(Ravenel, Devinatz Hopkins-Smith)

$\pi_0(E_n) = \boxed{W(k)[[u_1, \ldots, u_{n-1}]]}$

[illegible]

phase for

ring.

~~Top version:~~

~~Q:~~

Q: How to produce a condensed E_∞-ring spectrum E_n s.t. $\pi_0 E_n = W(k)[[u_1, \ldots, u_{n-1}]]$ holds as cond. ab gps?

Top version: How to produce $\mathbb{C}_p$ as a condensed ring?

2 answers:

1) $\mathbb{C}_p = \lim$

$$1) \quad \mathbb{Q}_p = \varprojlim_n \mathbb{Z}/p^n \ell$$

discrete

in Cond Rings,

$$= \left(\mathbb{Z}_p^\delta \right)^\wedge_p$$

in the condensed world.

$$2) \quad S \mapsto W\big(C(S, \mathbb{F}_p)\big)$$

perfect $\mathbb{F}_p$-algebra

unique flat $\mathbb{Z}_p$ algebra lifting $C(S, \mathbb{F}_p)$.

$$C(S, \mathbb{Z}_p) = \varprojlim_n C(S, \mathbb{Z}/p^n\mathbb{Z})$$

Likewise for F :

[illegible]

// It turns out that for spectra, there's more than just p you can intrinsically complete at.

There's a whole sequence

$$p, v_1, v_2, \dots$$

$p \in \pi_0 \mathbb{S} = \mathbb{Z}$

$\in [\Sigma^d \mathbb{S}/p, \mathbb{S}/p]$ acts in a degree-shifting way on X/p

for any $X \in S_p$

$v_2 : \Sigma^d X/(p, v_1) \to X/(p, v_1)$

...

$\rightsquigarrow$ You can complete any spectrum at $(p, v_1, \dots, v_{n-1})$.

(formally, Bousfield loc. wrt $S/(p, v_1, \dots, v_{n-1})$)

up to multiplying by a power of v,

$p, v_0, v_1, \dots, v_{n-1}$

$\in \pi_0 E_n$

~~Are~~ ~~cut~~ ~~out~~
the
~~have~~ ~~TT~~ Scurete
an open ideal

$\Rightarrow$ cpletn of
them is the
same CS cpletn
of $\overline{(I)}$.

Option 1:

$$\boxed{\left(E_n^\delta\right)^{\wedge}_{p,v_1,\dots,v_{n-1}} \quad \text{in the condensed world.}}$$

E_n^δ

or $\left(E_n^\delta\right)^{\wedge}_{p,v_1,\dots,v_{n-1}}$

~~dont~~ working in

E_n^δ-modules in

Compact Sp.

Option 2:

Lurie: $(k, G) \rightsquigarrow E(k,G)$

works also for $W(k)[[v_1,\dots,v_{n-1}]]$

k a perfect ring

& G a formal gp of ht $= n$

$$\boxed{E_G}(S) = E\Big(C(S,k), \text{pullback of } G\Big)$$

perfect ring

$$E_n^{\otimes} \longrightarrow E_n$$

both are $(p, v_1, \dots, v_{n-1})$-complete,

so can check

mod $p, v_1, \dots, v_{n-1}$

$\Rightarrow$ both are discrete,

hence Iso.

$$L_n S \in \langle E_n, E_n \otimes E_n, \dots \rangle$$

$$\big\downarrow \widehat{(p, v_1, \dots, v_{n-1})}$$

$$L_{K(n)} S \in \langle E_n \rangle$$

Can be more

precise about

how to build

$L_{K(n)} S$ from E_n.

$k = \overline{\mathbb{F}_p}$, G = formal gp of ht n.

$\exists$ a ... scheme / ...

$\Rightarrow$ group scheme / $\mathbb{F}_p$

parametrizing automorphisms

pro-constant

of pair (k, G)

$$\begin{array}{ccc} G & \xrightarrow{\cong\,\varphi} & G \\ | & & | \\ k & \xrightarrow{\cong} & k \end{array}$$

$$= \underline{\mathcal{G}},$$

$\mathcal{G}$: (Extended) Morava stabilizer gp

$$\mathcal{O}_{D_x}^{\times} \to \mathcal{G} \to \underline{\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)} = \hat{\mathbb{Z}}$$

units in ring of integers of division alg of inv. $1/n$

$/ \mathbb{Q}_p$

Eg G_{E_n}

$$\uparrow$$

$$\mathbb{S}$$

$$L_{K(n)}\mathbb{S} \xrightarrow{\sim} \text{“} E_n^{hG} \text{”}$$

Problem: for this to be true, it needs to be continuous htpy fixed pts on right.

$\vert$ $\mathcal{P}($ ... $)$

$$\boxed{\begin{gathered} H^{i}_{cont}(\mathcal{G}, \pi_* E_n) \\ \Downarrow \\ \pi_{*-p} L_{K(n)} S \end{gathered}}$$

$\mathcal{G} \curvearrowright \pi_0 E$

is <u>not</u> continuous
wrt discr top
on $\pi_0 E$
& prof top on $\mathcal{G}$.

What does
it mean that
$\mathcal{G}$ acts on E_n

$\mathcal{G}$: condensed gp

E_n: condensed spectrum

$\mathcal{G} \curvearrowright A$

cond ab gp

: $\mathcal{G} \times A \to A$

cond sets

s.t.

~~that~~ on S-valued pts,

get

$\mathcal{G}(S) \times A(S) \to A(S)$

group action on ab gp.

Could say same thing:

$$\mathcal{G} \to \underbrace{\underline{\mathrm{Act}}}_{\text{condensed gp anima}}(\mathrm{En})$$

Different perspective:

use slice topos

$\mathrm{CondAn}/B\mathcal{G}$

$\mathcal{G} \circlearrowright$
$$\begin{tikzcd} X_{\text{underlying}} \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & B\mathcal{G} \end{tikzcd}$$

$=$ sheaves of anima on $\text{extr disc}/B\mathcal{G}$

S, $\mathcal{G}$-torsor on S.

S extr disc profin set,

$$\begin{array}{l} X \\ \downarrow\ \mathcal{G}\text{-torsor} \\ S \end{array}$$

$$\begin{array}{l} X \\ \downarrow\ \mathcal{G}\text{-torsor} \end{array}$$

$\Longrightarrow$ $\downarrow$

$\underline{S}$

$\underline{S} =$ const. gp scheme

$= \mathrm{Spec}(C(S,k))$

$\underline{S} \longrightarrow BG$
$\quad\quad\quad\quad \shortparallel$
stack of formal gps Iso to G_0

$\Longrightarrow$ classifies a new formal gp G' on $\underline{S}$

$\Longrightarrow$ $E(C(S,k), G')$

How to check

$$\boxed{L_{K(n)} S \xrightarrow{\sim} E_n^{hG} ?}$$

T

condensed spectrum.

$$L_n S \wedge$$

$p_1 \sim V_{n-1}$

$\mathbb{G}$

$$\begin{tikzcd} E_n \arrow[r] \arrow[d] & \widetilde{E_n} \arrow[d] \\ * \arrow[r, hook] & B\mathbb{G} \end{tikzcd}$$

take sections

$\Rightarrow$ cond spectrum

$$\boxed{E_n^{h\mathbb{G}}}$$

Note $E_n^{h\mathbb{G}}$ is

K(n)-local:

$$E_n^{h\mathbb{G}} = \varprojlim \left(E_n \to C(\mathbb{G}, E_n) \rightrightarrows C(\mathbb{G}\times\mathbb{G}, E_n) \cdots \right)$$

enough to check

mod $p, v_1, \dots, v_{n-1}$

$$L_n S/_{p, v_1, \dots, v_{n-1}} \to E_n/_{p, \dots, v_{n-1}}^{hG}$$

$\mathrm{Can}(- \otimes E_n)$

check on

$$E_n/_{p, \dots, v_{n-1}} \xrightarrow{\sim} (E_n \otimes E_n/_{p, \dots, v_{n-1}})^{hG}$$

$$C(G, E_n/_{p, \dots, v_{n-1}})^{hG}$$

$$\|$$

$$E_n/_{p, \dots, v_{n-1}}$$

Rk:

⚠ E_n is

Spi[

$$E_1 = KU^\wedge_p$$

$E_1 \otimes E_1$ not p-complete!

(E_1 non-connective)

$L_n S \qquad E_n \qquad E_n \otimes E_n$

Rk: We just talked about condensed spectra w/ Γ-action,

Γ = profinite gp.

What's the relation
w/ previous
approaches?

$(E_n^{hI})^{\wedge}_{?}$ $L_{K(n)} \mathbb{S} =$ "E_n^{hI}"

<u>Devinatz-Hopkins:</u>

Produced a hyper-sheaf
of E_∞-ring spectra
on the site of
finite continuous
$\mathbb{I}$-sets
s.t.:

1) on $*$, it's
$L_{K(n)} \mathbb{S}$

2) Have descent SS

$H^i_{\mathrm{cont}}(I; \pi_* E_n[-])_{\mathrm{cons}}$

3) stalk @ pt of this site

$$= \lim_{U \subset I} \text{"}E_n^{hU}\text{"}$$

is something whose $K(n)$-localization is E_n.

Thm X discrete anima.

2 ways of saying $I \curvearrowright X$:

1) view X as condensed, take what we discussed

2) Find a sheaf on finite cont's I-sets w/ stalk X.

<u>Claim:</u> ∞-cat of

n-truncated discrete cond anima w/ action of I

$\simeq$ <u>Postnikov completion</u>

$\boxed{\text{of } \infty\text{-cat of sheaves of anima on site of finite } I\text{-sets}}$ n-truncated

$$\simeq \varprojlim \left(Sh(*)_{\leq n} \rightrightarrows Sh(I)_{\leq n} \rightrightarrows Sh(I \times I)_{\leq n} \right)$$

sheaves of
anima on
top space X.

Rk: if $\mathrm{hcd}(X) < \infty$

(e.g. $X = |\mathcal{G}|$),

then Postnikov
completion
= hypersheaves.

Rk: This is one
setting where
the condensed
approach is the
only "reasonable"
one:

transchromatic

htpt theory.

S

$\boxed{L_{K(n)}S} \neq E_n^{hG}$

$\boxed{L_{K(n-1)}S}$

L_nS

$\vdots$

$L_{K(0)}S$

$$L_{K(n-2)} L_{K(n-1)} E_n$$

$$\pi_0 L_{K(n-1)} E_n$$

$$= \cancel{W(k)[[u_1, \dots, u_{n-2}]]}$$

$$\mathrm{dia}\,[W(k)[[u_1, \dots, u_{n-1}]]$$

$[v_{n-1}^{-1}]$

$\beta_1 v_{i-1}^{2nd}$

is not a top ring,

completeness

to study classically.

$K(E_n)^{\otimes}$

(Blumberg-Mandell, Barwick-Lawson)

$\boxed{k : \text{finite field}}$

$K(E_n{}_{\geq 0})^{\otimes} = \varprojlim K(\dots)$

$K(\pi_0 E_n)^{\otimes} = \varprojlim K(\dots)$