Solid $\mathbb{Q}_p$-modules

$$\parallel$$

$$\mathrm{Mod}_{\mathbb{Q}_p}(\mathrm{Solid}_{\mathbb{Z}})$$

$$\overset{\mathbb{Z}_p[\frac{1}{p}]}{\mathbb{Q}_p} \otimes \overset{\mathbb{Z}_p[\frac{1}{p}]}{\mathbb{Q}_p} = \mathbb{Q}_p$$

$$\Rightarrow \mathrm{Mod}_{\mathbb{Q}_p}(\mathrm{Solid}_{\mathbb{Z}}) \cong \mathrm{Solid}_{\mathbb{Q}_p}$$

$$M \otimes_{\mathbb{Q}_p} N = M \otimes N .$$

$$\mathbb{Z}_p \otimes \mathbb{Z}_p = \mathbb{Z}_p \quad \Big| \quad \mathbb{Z}_p = \mathbb{Z}[[T]]/_{T-p}$$

$$\mathbb{Q}_p \otimes \prod_I \mathbb{Z}$$

$$= \left(\mathbb{Z}_p \otimes \prod_I \mathbb{Z}\right)\left[\tfrac{1}{p}\right]$$

$$= \left(\boxed{\prod_I \mathbb{Z}_p}\right)\left[\tfrac{1}{p}\right]$$

$\nearrow$

cpct proj in $\mathrm{Solid}_{\mathbb{Q}_p}$.

"Smith Spaces"

VS

Banach spaces

$$\cong \Big(\bigoplus_I \mathbb{Z}\Big)^{\wedge}_{p}\Big[\tfrac{1}{p}\Big]$$

There are two relationships between Smith spaces & Banach spaces

1) Duality:

i) $\underline{\mathrm{Hom}}_{\mathbb{Q}_p}\Big(\prod_I \mathbb{Z}_p\big[\tfrac{1}{p}\big], \mathbb{Q}_p\Big)$

$$= \Big(\bigoplus_I \mathbb{Z}_p\Big)^{\wedge}_{p}\Big[\tfrac{1}{p}\Big]$$

$(C(S, \mathbb{Q}_p)$ if $\big(\prod_I \mathbb{Z}_p\big)\big[\tfrac{1}{p}\big] = \mathbb{Z}[S]^{\bullet} \otimes \mathbb{Q}_p$

ii) $\underline{\mathrm{Hom}}_{\mathbb{Q}_p}\Big(\big(\bigoplus_I \mathbb{Z}_p\big)^{\wedge}_{p}\big[\tfrac{1}{p}\big], \mathbb{Q}_p\Big)$

$$= \underline{\mathrm{Hom}}_{\mathbb{Q}_p}\Big(\big(\bigoplus_I \mathbb{Z}\big)^{\wedge}_{c}, \mathbb{Z}_p\Big)\Big[\tfrac{1}{p}\Big]$$

cts map of Banach

$$= \Big(\prod_I \mathbb{Z}_p\Big)\Big[\tfrac{1}{p}\Big]$$

spaces is
bounded.

⚠ i) holds for $\underline{\mathrm{RHom}}$, i.e.

$\underline{\mathrm{Ext}}^i = 0$
for $i>0$.

but ii)
Does not??
(probably independent
of ZFC)

---

2) Banach space =
~~uncount~~ $>\aleph_0$ filtered colim
of Smith spaces,

Smith space =
filtered lim
of Banach
spaces.

$$\hookrightarrow \left(\bigoplus_I \mathbb{Z}\right)^\wedge_p\left[\tfrac{1}{p}\right]$$

$$\|$$

$$\bigcup_{\substack{f: I \to \mathbb{R}_{\geq 0} \\ \text{tend to} \\ 0}} \left(\prod_{i\in I} (\mathbb{Q}_p)_{\leq f(i)}\right)\left[\tfrac{1}{p}\right]$$

$\uparrow$ $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p$

$$\subsetneq \prod_I \mathbb{Z}_p\left[\tfrac{1}{p}\right]$$

Note: Smith spaces are not closed under coker.

Claim: quotients of Smith spaces $\circlearrowleft$

$=$ compact objects in $\text{Solid}_{\mathbb{Q}_p}$

$= \boxed{C\left[\tfrac{1}{p}\right],}$

$\prod \mathbb{Z}_p \leftarrow \prod \mathbb{Z} \to C$: pro-p-abelian gp.

form an abelian subcategory.

qSmith is Smith $\Leftrightarrow$ qseparated.

"non-separated Smith spaces".

$$\left(\prod_n \mathbb{Z}/p^n\mathbb{Z}\right)\left[\tfrac{1}{p}\right]$$

$\exists$ analog for Banach as well:

qBanach = $M \in \mathrm{Solid}_{\mathbb{Q}}$ s.t. $M(*)$ $M[\frac{1}{p}]$,

$\cong$

$\{$derived p-complete ab gps$\}$/isog

$M \in \mathrm{Solid}_{\mathbb{Q}}$ s.t. $M(*) \to M$ inducing $\pi_0(M(*)^\wedge_p) \xrightarrow{\sim} M$

defined

Rk: $\mathbb{Q}$ Separated Solid$_{\mathbb{Q}}$

= unions of Smith spaces.

& all such are flat, collection of them is closed under $\otimes$.

small projs $\overset{\rightarrow}{\otimes}$ M

$$\prod \mathbb{Z}_p[\tfrac{1}{p}] \overset{L}{\otimes} C[\tfrac{1}{p}]$$

$$= \left(\prod \mathbb{Z}_p \overset{L}{\otimes} C\right)[\tfrac{1}{p}]$$

$$= \prod C[\tfrac{1}{p}]$$

$$\mathrm{Tor}^i_{\mathbb{Q}}(M,N) = 0 \quad i > 1$$

$M, N \in \mathrm{Solid}_{\mathbb{Q}_p}$

Aim: Frechet Spaces V, W

$$\left(V \cong \varprojlim_n V_n \right.$$

Banach

along dense transition maps

$$V \otimes W = V \overset{\pi}{\hat{\otimes}} W$$

projective tensor product

$$\varprojlim_n V_n \overset{\pi}{\hat{\otimes}} \varprojlim_m W_m$$

$$= \varprojlim_{n,m} V_n \overset{\pi}{\hat{\otimes}} W_m$$

tensor product

of Banach spaces:

$$\left(\left(\bigoplus_I \mathbb{Z}\right)_p^\wedge[\tfrac{1}{p}]\right) \overset{\pi}{\hat\otimes} \left(\left(\bigoplus_J \mathbb{Z}\right)^\wedge_p[\tfrac{1}{p}]\right)$$

$$\left(\bigoplus_{I\times J} \mathbb{Z}\right)^\wedge_p[\tfrac{1}{p}].$$

Basic calculation:

$$\left(\bigoplus_I \mathbb{Z}\right)^\wedge_p[\tfrac{1}{p}] \otimes \left(\bigoplus_J \mathbb{Z}\right)^\wedge_p[\tfrac{1}{p}]$$

||

$\cup$ $\qquad\qquad$ $\cup$

$f: I \to \mathbb{R}_{\geq 0}$ tending to 0 $\qquad$ $g: J \to \mathbb{R}_{\geq 0}$

Need: any function

$h: I\times J \to \mathbb{R}_{\geq 0}$

tending to 0

is dominated by some

$(f\otimes g)(i,j) = f(i)\cdot g(j)$

$(f \times g)(i,j)$

$$f(i) = \sqrt{\max_j h(i,j)}$$

$$g(j) = \sqrt{\max_i h(i,j)}$$

True for Banach ✓

$$\begin{tikzcd}
\prod_{\mathbb{N}} \mathbb{Q}_p \otimes \prod_{\mathbb{N}} \mathbb{Q}_p \arrow[d, equal] \arrow[r, phantom, "\simeq"] & \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Q}_p \arrow[d, equal] \\
\bigcup_{f:\mathbb{N}\to\mathbb{R}_{\geq 0}} \left(\prod_{n\in\mathbb{N}} (\mathbb{Q}_p)_{\leq f(n)}\right)[\tfrac{1}{p}] & \bigcup_{g:\mathbb{N}\times\mathbb{N}\to\mathbb{R}_{\geq 0}}
\end{tikzcd}$$

Need: $h : \mathbb{N}\times\mathbb{N} \to \mathbb{R}_{\geq 0}$

↳ dominated by

Some $f \times g$, ~~f~~

Take $f(n) = \max\{1, \max_{i,j \leq n} h(i,j)\}$

$\quad \|$

$q(a) \quad i,j=n$

$\mathfrak{N}$

We'll also need:

$$\underset{\text{Banach}}{V} \otimes \prod_{\mathbb{N}} \mathbb{Q}_p$$

$$= \prod_{\mathbb{N}} V .$$

This is the most subtle

⚠ fails on $\mathbb{Z}_p$-mod:

$$\left(\bigoplus_I \mathbb{Z}\right)^\wedge_p \otimes \prod_{\mathbb{N}} \mathbb{Z}_p$$

$$\neq \prod \left(\bigoplus_I \mathbb{Z}\right)^\wedge_p$$

(reduce mod p)

I'll give an argument using concept of <u>nuclearity</u>.

Note: A smith space

$$V = \left(\prod \mathbb{Z}_p\right)\left[\tfrac{1}{p}\right]$$

can also be given a "Banach topology":

$$V^{B} = \left(\left(\prod \mathbb{Z}_p\right)^{\delta}\right)^{\wedge}_{p}\left[\tfrac{1}{p}\right].$$

In fact, $V \mapsto V^{B}$ is natural in V, we have map

$$V^{B} \xhookrightarrow{c} V$$

$$W \xrightarrow{f} V$$

Q: when does it factor thru V^B?

A: · f is dual to a cpct operator on Banach spaces

· f is trace-class.

Def: A trace-class map $W \to V$ is a map coming from some element of

$$\boxed{\left|(\underline{\mathrm{Hom}}(W, \mathbb{Q}_p) \otimes V)(* \right|}$$

via the natural map ~~contraction~~

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(W, \mathbb{Q}_p) \otimes V \arrow[d] \\
\underline{\mathrm{Hom}}(W, V)
\end{tikzcd}$$

~~Reformulation~~

Reformulation: $V \in \mathrm{Solid}_{\mathbb{Q}_p}$.

Define a condensed $\mathbb{Q}_p$-module ~~[illegible]~~ V^{tr} by

$$V^{tr}(S) = \left(C(S, \mathbb{Q}_p) \otimes V \right)(*).$$

Then $\exists$ natural map

$$V^{tr} \to V,$$

& for W smooth space, a trace-class map

$W \longrightarrow V$

$\Leftrightarrow$ map factoring thru V^{tr}.

$V \mapsto V^{t}$

functor $\mathrm{Solid}_{\mathbb{Q}_p} \to \mathrm{Solid}_{\mathbb{Q}_p}$

$V^{tr} \to V$

<u>Claim:</u> V Smith space

$$\begin{tikzcd} V^{tr} \arrow[r, phantom, "="] \arrow[dr] & V^{B} \arrow[d] \\ & V \end{tikzcd}$$

<u>Pf:</u> $C(S, \mathbb{Q}_p) \otimes (\prod_I \mathbb{Z}_p)(\frac{1}{p})$

$= (C(S, \mathbb{Z}_p) \otimes \prod_I \mathbb{Z}_p)[\frac{1}{p}]$

let's use fact that

solid p-complete $\mathbb{Z}$s

are closed under

$\otimes$.

$$= \left(\varprojlim_n \underbrace{\left(C(S,\mathbb{Z}/p^n) \otimes_{\mathbb{Z}/p^n} \prod \mathbb{Z}/p^n\right)}_{\delta}\right)\left[\tfrac{1}{p}\right]$$

$$= \left(\varprojlim_n C(S,\mathbb{Z}/p^n) \otimes \left(\prod \mathbb{Z}/p^n\right)^{\delta}\right)\left[\tfrac{1}{p}\right]$$

$$= C\left(S, \left(\left(\prod \mathbb{Z}_p\right)^{\delta}\right)^{\wedge}_{p}\left[\tfrac{1}{p}\right]\right)$$

$$= C(S, V^B).$$

Nuclearity

Def: $M \in \mathrm{Solid}_{\mathbb{Q}_p}$ is

nuclear $\iff$

$\forall$ extr disc S,

$$\left(C(S,\mathbb{Q}_p) \otimes M\right)(*) \xrightarrow[\sim]{} \underline{M}(S)$$

Equivalently, $M^{tr} \xrightarrow{\sim} M$

Rk: both sides are exact, colim. pr. functors of M

(flatness of $C(S, \mathbb{Q}_p)$)

$\Rightarrow$ $\mathrm{Nuc}_{\mathbb{Q}_p}$ is an abelian subcategory closed under colimits.

Moreover, if we were to make the derived analog of this definition,

derived nuclear $\Leftrightarrow$ each π_i is nuclear.

Thm: $M \in \mathrm{Solid}_{\mathbb{Q}_p}$ is nuclear

(⟸) M is a filtered colimit of qBanach spaces.

(nuclear = generated under colims by Banach spaces.)

it's not an Ind-category.

i.e. not cptly generated.

If is N_1-compactly generated,

but the N_1-compact object are also different from the Banach spaces!

N_1-compact objects

~ cokernels of

"Basic nuclear modules"

=

$$\underset{\underset{n}{\longrightarrow}}{\mathrm{colim}}\left(V_0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \cdots\right)$$

↑ trace class maps

↑ cpt proj

<u>Rk:</u> $\mathrm{Nuc}_{\mathbb{Q}_p}$ is independent of $\kappa > \aleph_0$

(every Banach space is κ-condensed)

$$\|$$

$$\bigcup \text{ Banach spaces } \left(\bigoplus_{\mathbb{N}} \mathbb{Z}\right)^{\wedge}_{p}\left[\tfrac{1}{p}\right]$$

<u>Pf:</u> First suppose M is Nuclear.

write

$$M = \varinjlim_{i \in I} M_i ,$$

M_i compact objects
in Solid Sp.

these " = q Smith spaces

Note that

$M \mapsto M^{tr}$

commutes w/
colimits)

~~So enough to~~
note that

$\Rightarrow \quad M \simeq \varinjlim_i M_i^{tr}$

$S_{1i} \to S_{0i} \to M_i \quad$ q Smith

$\Rightarrow \quad M_i^{tr} = $ q Banach
as

$(\text{Smith})^{tr} = \text{Banach}.$

For the course, it
suffices to show
that a Banach space

is nuclear.

$$(C(S,\mathbb{Q}_p)\otimes V)[0] \xrightarrow{\cong} V(S)$$

$\uparrow$

tensor product of Banach spaces is usual tensor product.

(choose basis & calculate)

Cor: $\mathrm{Nuc}_{\mathbb{Q}_p} \subset \mathrm{Solid}_{\mathbb{Q}_p}$ is also closed under $\otimes$.

Pf: Banach $\otimes$ Banach = Banach.

This notion of nuclearity is in general cleanest in derived context.

$(\mathcal{C}, \otimes)$ ← commut colimits

↑

cpctly gen stable
∞-cat

1 cpct object.

Some of these facts
hold in this context,
others not.

e.g. Nuc N, - cpctly
gen,
closed under
$\otimes$ ✓

Surprising fact:

$$\mathrm{Nuc}_{\mathbb{Q}_p} \subset \mathrm{Solid}_{\mathbb{Q}_p}$$

is closed under
countable products!

Rk: This fails for $\mathbb{Z}_p$

$\mathrm{Nuc}_{\mathbb{Z}_p}$ = filtered colim of derived p-completions of discrete gps

$\ni \mathbb{Z}_p$

but $\prod_{\mathbb{N}} \mathbb{Z}_p$ is not nuclear.

Key case:

$$\prod_{\mathbb{N}} \mathbb{Q}_p \in \mathrm{Nuc}_{\mathbb{Q}_p}.$$

Pf: $$\prod_{\mathbb{N}} \mathbb{Q}_p = \bigcup_{f:\mathbb{N}\to\mathbb{R}_{\geq 0}} \Big(\prod_{n\in\mathbb{N}} \mathbb{Q}_{p,\leq f(n)}\Big)$$

$\uparrow$

Small

spans

Claim: can replace

$$\left(\prod_{n\in\mathbb{N}} (\mathbb{Q}_p)_{s\in(n)} \right) \left[\frac{1}{p}\right]$$

by Banachification & this doesn't change filtered union.

Suffices to see that transition maps eventually are trace class.

$$f \in f'$$

$$f'(n) = p^n \cdot f$$

$$\prod_{\mathbb{N}} \mathbb{Z}_p\left[\tfrac{1}{p}\right] \longrightarrow \prod_{\mathbb{N}} \mathbb{Z}_p\left[\tfrac{1}{p}\right]$$

$\cdot p^n$ in

n^{th} coordinate

= basic example
of a cpct operator
"tr class."

General case:

$$\prod_{n\in\mathbb{N}} M_n$$

write each M_n =
filtered colim
of q Banach

$$\Rightarrow \prod_{n\in M} M_n = \mathcal{W}$$

filtered
colim

if $\prod$ q Banach

(Grothendieck AB6)

$\prod$ exact, so that reduces to $\prod$ Banach.

~~But any collection of Banach spaces = Retn~~

Need:

$$\prod_{\mathbb{N}} \underset{\substack{\uparrow \\ \text{Banach}}}{V}$$

is nuclear.

~~But this~~

<u>Claim:</u> $\prod_{\mathbb{N}} V \simeq \left(\prod_{\mathbb{N}} \mathbb{Q}_p\right) \otimes V$

Pf: $V = C(S, \mathbb{Q}_p)$

II. $V = C(S, \mathbb{Q}_p)$

By nuclearity of $\prod_{\mathbb{N}} \mathbb{Q}_p$, have

$$V \otimes \prod \mathbb{Q}_p = C(S, \prod_{\mathbb{N}} \mathbb{Q}_p)$$

$$= \prod C(S, \mathbb{Q}_p).$$

$$\left(C(S, \mathbb{Q}_p) \otimes M\right)(*) \xrightarrow{\sim} \underline{C}(S, M)(*)$$

$$\Rightarrow \quad C(S, \mathbb{Q}_p) \otimes M \xrightarrow{\sim} C(S, M)$$

because cpct proj are closed under $\otimes$.

Addendum: On $\mathrm{Nuc}_{\mathbb{Q}_p}$,

$$V \otimes -$$

commutes w/ countable products if V is Banach.

Pf: Same idea reduce to

$$V \otimes \prod_{\mathbb{N}} W$$

$$\simeq \prod_{\mathbb{N}} V \otimes W \, .$$

For that, use

$$V \otimes \prod_{\mathbb{N}} W$$

$$\simeq (V \otimes W) \otimes \prod_{\mathbb{N}} \mathbb{C}$$

$$\simeq \prod_{\mathbb{N}} V \otimes W \, . \qquad \blacksquare$$

Come back to Frechet spaces:

Lemma: (Mittag-Leffler):

if $(V_n)_{n \in \mathbb{N}}$ is

a projective system of Banach spaces w/ dense transition maps, then

$$R\varprojlim^1 V_n = 0 .$$

(in $\mathrm{Cond}_{\mathbb{Q}_p}$)

$\Rightarrow$

$$0 \to \varprojlim_n V_n \to \prod_n V_n \xrightarrow{1-\sigma} \prod V_n \to 0 .$$

in $\mathrm{Solid}_{\mathbb{Q}_p}$.

reduce to:

$$\prod_n V_n \otimes \prod_m W$$

$$\simeq \prod_{n\times m} V\otimes W$$

$$\prod_n V \simeq V \otimes \prod_n \mathbb{Q}_p$$

$$\prod_n W \simeq W \otimes \prod_n \mathbb{Q}_p$$

Descent

A : comm algebra in $\mathrm{Nuc}\,\mathbb{Q}_p$

Define $\mathrm{Nuc}_A :=$

$\mathrm{Mod}_A(\mathrm{Nuc}\,\mathbb{Q}_p)$.

Rk: Could also look at $\mathrm{Mod}_A(\mathrm{Solid}_{\mathbb{Q}_p})$, & redo the nuclear discussion fine.

Result would be the same.

<u>Thm:</u> $A \to B$

map of comm. algebras in $\mathrm{Nuc}_{\mathbb{Q}_p}$.

Suppose:

1) $- \otimes_A B : \mathrm{Nuc}_A \to \mathrm{Nuc}_B$

has finite tor-dimension

~~2) $M \in \mathrm{Nuc}_A$.~~

~~If~~ $M \otimes_A B$

$\mathcal{N}_A = \mathrm{Mod}_A(\mathrm{DNuc}_{\mathbb{Q}_p})$

as ∞-category

2) $M \in \mathcal{N}_B$.

If $M \underset{A}{\otimes} B$ B connective,

then so is M.

Then have descent:

$$\mathcal{N}_A \xrightarrow{\sim} \varprojlim_{\Delta} \left(\mathcal{N}_B \rightrightarrows \mathcal{N}_{B \underset{A}{\otimes} B} \rightrightarrows \cdots \right)$$

& furthermore,

$$\mathrm{Perf}(A(\!(x)\!)) \xrightarrow{\sim} \varprojlim \left(\mathrm{Perf}(B(\!(x)\!)) \rightrightarrows \mathrm{Perf}(B \underset{A}{\otimes} B(\!(x)\!)) \rightrightarrows \cdots \right)$$

& (~~if each $B\otimes$~~ B/A flat)

$$\mathrm{Vect}(A(\!(x)\!)) \xrightarrow{\sim} \varprojlim \left(\mathrm{Vect}(B(\!(x)\!)) \rightrightarrows \mathrm{Vect}(B \underset{A}{\otimes} B(\!(x)\!)) \right)$$

if A, B are "Fredholm"

ex: $A \longrightarrow B$

map of p-torsionfree p-complete rings which is faithfully flat mod p

$\Rightarrow A_0[\frac{1}{p}] \to B_0[\frac{1}{p}]$ satisfies the hypothesis.

Conclusion = Thm of Drinfeld [Vect]

Mathew [Perf].

Sketch of proof:

Use Barr-Beck:

need that

$$- \otimes_A B : N_A \to N_B$$

i) Construct

& Combine w/ geometric realizations.

Countable limit

$$\begin{tikzcd}
\varprojlim_{n} \left(\varprojlim_{\Delta_{\le n}} X \right) \arrow[d] \\
\boxed{\prod_{n}} \quad \boxed{\varprojlim_{\Delta_{\le n}} X} \arrow[d] \\
\prod \quad \varprojlim_{\Delta_{\le n}} X
\end{tikzcd}$$

$\bigotimes_A B$ = geom real. of

[illegible]

absolute $\otimes$

~~Nuc~~

General result: (in tensor cat)

Nuc ∩ Cpct

||

dualizable under ⊗.

$$X \xrightarrow{id} X$$

has tr. class rep.

$$1 \xrightarrow{\text{coev}} \mathbb{D}X \otimes X$$

& is generated

st

$$\mathrm{cone}\left(M \xrightarrow{1-f} M\right)$$

$\uparrow$

cpct projector,

f trace-class.

<u>Claim:</u> (Ex): $\mathbb{Q}_p$

ϕ is <u>Fredholm</u>:

Φ

and such

$$\left(M \xrightarrow{\phi_n 1-f} M\right) = X$$

is <u>discrete</u>

(relate to $\mathbb{Q}_p$:

$$X(S) \otimes_{\mathbb{Q}_p^{\delta}} \mathbb{Q}_p$$

$\implies$ it's a perfect complex of $\mathbb{Q}_p$