Complex Analysis

__Peter:__ $0 < p \leq 1 \rightsquigarrow$

$$\mathrm{liq}_p \subseteq \mathrm{Cond}_{\mathbb{R}}$$

abelian category

w/ $\otimes^{\mathrm{liq}_p}$

compact projectives

$\mathcal{M}_{<p}[S]$,

S extr. disc.

__Q:__ ~~Explain~~ Are liquid modules

"complete enough"?

__Test:__ Calculate some completed tensor products.

$\mathcal{M}$-complete $\subset \mathrm{liq}_p \ \forall p$

Ex: V, W Banach

Ex! V, W Banach space,

$V \overset{\text{lin,}\,\epsilon_p}{\otimes} W$ is

not a Banach space

(not loc. convex)

(less complete than $V \overset{\pi}{\otimes} W$)

Same if V & W are Smith space

?

By definition, S, T profinite

$$\mathcal{M}_{cp}[S] \overset{\text{lin}}{\otimes} \mathcal{M}_{cp}[T]$$

$$\simeq \mathcal{M}_{cp}[S \times T]$$

Special case: $S = \mathbb{N} \cup \{\infty\}$

$[\infty]$ ...

$\mathbb{R} \longrightarrow M_{<p}(\ldots)$

$\downarrow$

$\mathbb{R}$

$\Rightarrow$ summand

$$\ell^{<p}_{\text{smith}} : \quad \text{smith } \ell^{<p}\text{ space, an analog of usual sequence space}$$

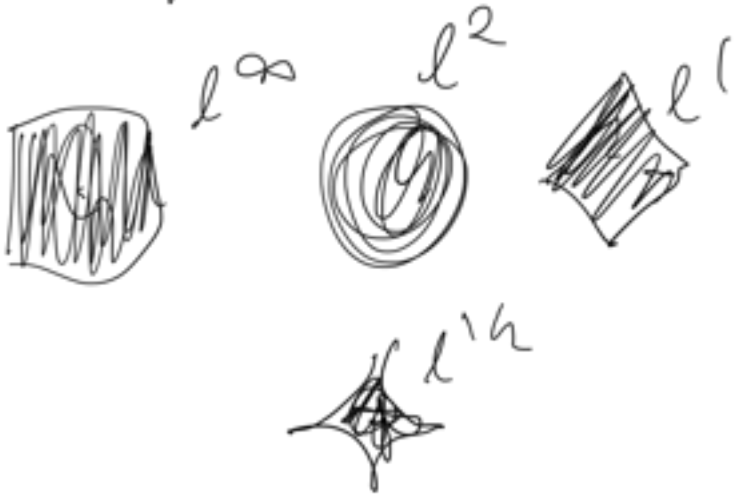
$$= \left\{ (\lambda_n)_{n\in\mathbb{N}} \;\middle|\; \sum |\lambda_n|^{p'} < \infty, \text{ some } p' < p \right\} \subset \mathbb{R}^{\mathbb{N}}$$

= "free $\mathbb{R}_p$-vector space in a null-convergent sequence"

Want to recall an idea of Grothendieck

in functional analysis.

$0 < p \le 1 \quad 1 \le p < \infty$



These subtleties are important in the nuclear context.

Recall: there is a notion of "trace-class" or "1-summable" maps of Banach spaces

$$V \to W.$$

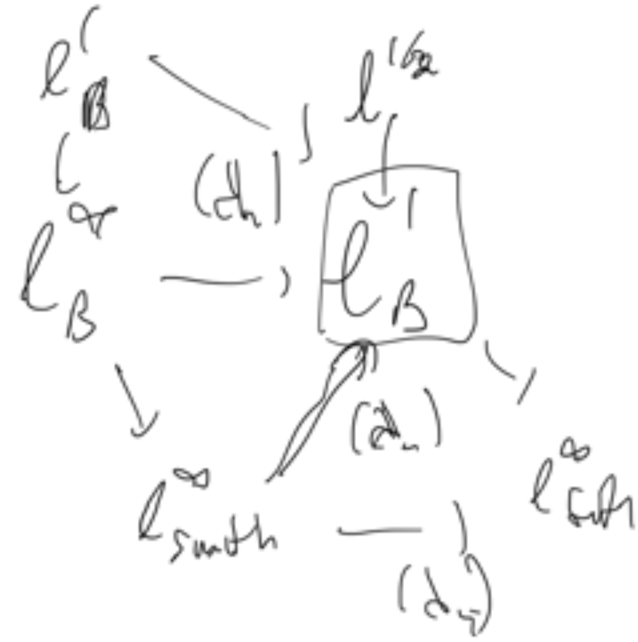
ex: $\ell^\infty \xrightarrow{(a_n)} \ell^1$

$\sum |\lambda_a| < \infty$

any trace class factors

Mystery:

Rk:



are unstable?

Rk: any p-summable map is p'-summable for p' > p;

composite of 2 1p-summable maps

it's ½-summable.

$0 < p, q < \infty$

✓

commensurable after finite composites.

<u>Def</u>: • A <u>DNF</u>-space is a (segin?) filtered union of Banach spaces along trace-class injections.

• A <u>NF</u> is a countable inverse limit of Banach spaces along dense trace-class maps.

<u>Above discussion</u>:

You get the same class of spaces if

class of spaces to
you do the analogous
thing the p-Banach
for any p.

<u>Claim:</u> V, W DNF spaces

$\Rightarrow$

$$V \otimes^{\text{lim}_p \text{la}} W = V \hat{\otimes} W$$

(completed tensor product of Grothendieck)

(use $V = \bigcup_n \ell^{<p}_{\text{Sch}}$)

<u>Rk:</u> Same is true for NF spaces, but this is harder to see.

<u>Ex:</u> Rings of holomorphic

rings.

$\bar{D}$: closed unit disk $\subseteq \mathbb{C}$

$\mathcal{O}(\bar{D})$ = f'ns holomorphic on an open nbhood of $\bar{D}$.

DNF space

$$= \bigcup_{\varepsilon > 0} \{ (c_n) \mid \textstyle\sum c_n r^n \text{ conv}, \ r = 1+\varepsilon \}$$

↑ Banach space

Sequence space, when ε shrinks the transition maps are given by diagonal matrix

$$(1, t, t^2, t^3, \ldots)$$

$\longrightarrow$ trace-class.

$$\mathcal{O}(\overline{D_1}) \overset{L}{\otimes} \mathcal{O}(\overline{D_2})$$

$$= \mathcal{O}(\overline{D_1 \times D_2})$$

↑ 2-dim'l polydisk.

Cor: $\overline{D}$, $\overline{D'}$ two closed disks in $\mathbb{C}$.

Then! contained in bigger $\overline{E}$.

Then $\mathcal{O}(\overline{D}) \overset{\text{liq}, L}{\otimes}_{\mathcal{O}(\overline{E})} \mathcal{O}(\overline{D'})$

$$= \mathcal{O}(\overline{D \cap D'})$$

Pf: First replace $\mathcal{O}(\overline{\ }$

with $\mathbb{C}[T]$

$$\mathcal{O}(\overline{D}) \overset{L}{\underset{\mathbb{C}[T]}{\otimes}} \mathcal{O}(\overline{D}')$$

$$= \mathcal{O}(\overline{D}) \overset{L}{\otimes} \mathcal{O}(\overline{D}') / (T_1 - T_2)$$

$$= \mathcal{O}(\overline{D} \times \overline{D}') / T_1 - T_2$$

$$= \mathcal{O}(\overline{D} \cap \overline{D}')$$

Special case:

$$\mathcal{O}(\overline{E}) \overset{L}{\underset{\mathbb{C}[T]}{\otimes}} \mathcal{O}(\overline{E})$$

$$= \mathcal{O}(\overline{E})$$

$\Rightarrow$ tensor over $\mathcal{O}(\overline{E})$ is same as over $\mathbb{C}[T]$.

Special case: $\bar{D} \cap \bar{D}' = \emptyset$

$$\Rightarrow \mathcal{O}(\bar{D}) \otimes_{\mathcal{O}(E)} \mathcal{O}(\bar{D}') = 0.$$

---

algebra reflects topology ~~of~~ $\mathbb{C}$.

Where are we going?

<u>Thm:</u> X cpct Riemann surface, E vector bundle $/X$. Then:

1) $H^i(X; E)$ is f.d. v.s. $\forall i$, 0 $i >> 0$ $(i \geq 2)$

2) $H^0(X; E)^\vee \cong H^1(X; \Omega^1 \otimes E^\vee)$

How we'll prove it:

Produce a theory of Quasi-coherent sheaves on cpct cplx manifolds w/ 6-functor formalism

(⟹ makes it easy to prove duality statements locally)

---

1) will come from:

$$\mathrm{Perf}(\mathbb{C}(x)) = \begin{matrix} \mathrm{Nuc}_{\lim_p} \\ \cap \\ \mathrm{cpct}_{\lim_p} \end{matrix}$$

2) can work locally.

A theory of q.coh

A [illegible] of [illegible]
sheaves of complex an space
<u>has</u> to be derived.

<u>Reason:</u> $- \otimes \mathcal{O}(D)$ is $\mathcal{O}(E)$ not exact.

<u>Pf:</u> $C_{-1} = \operatorname{coker}(\mathcal{O}(E) \overset{c}{\hookrightarrow} \mathcal{O}(D'))$

← non-Hausdorff quotient

$\otimes$ fuses up:

$\mathcal{O}(E) \hookrightarrow \mathcal{O}(D')$

becomes

$G(D) \to O$

Rk: We produced a G-functor formalism, solid QCoh for coherent cohomology in algebraic geometry

c.f. Condensed.pdf

generalized to idea of "analytic spaces", new idea of "open subset".

I'll take a half-classical approach, use usual underlying topological space, put a sheaf of condensed rings on it.

We'll build on the

"original" six-functor formalism,

$D(Sh(X;R))$

X: fin. dim'l locally cpct Hausdorff space.

Reminders on $Sh(X;R)$:

$$D(Sh(X;R)) = Sh(X; D(R)) = Sh^{hyp}(X; D(R))$$

2 perspectives:

1) take classical thy as cat, derive it

?) directly define the derived theory, derived analog of classical theory.

$$X \xrightarrow{f} Y$$

$$\Rightarrow \quad f^* : Sh(Y; D(\mathbb{Z})) \to Sh(X; D(\mathbb{Z}))$$

right adjoint

$$f_* : Sh(X, D(\mathbb{Z})) \to Sh(Y, D(\mathbb{Z}))$$

$$f_* \mathcal{F}(V \subset Y) = \mathcal{F}(f^{-1}V)$$

$\cap$
X

$f^* \mathcal{F}$ left adjoint:

pullback commutes w/ stalks

$$(f^* \mathcal{F})_x = \mathcal{F}_{f(x)}$$

$$\mathcal{F}_y = \varinjlim_{y \in V \subset Y} \mathcal{F}(V)$$

---

f^* const = const

f_* more subtle.

$$X \xrightarrow{p} pt$$

$$p_* \underline{\mathbb{Z}} \in D(\mathbb{Z})$$

is a complex

computing (sheaf) cohomology of X.

Might expect: $X \xrightarrow{f} Y$,

f_* is "in-families" analog of cohomology.

But no! ~~Cohe~~

~~fact~~

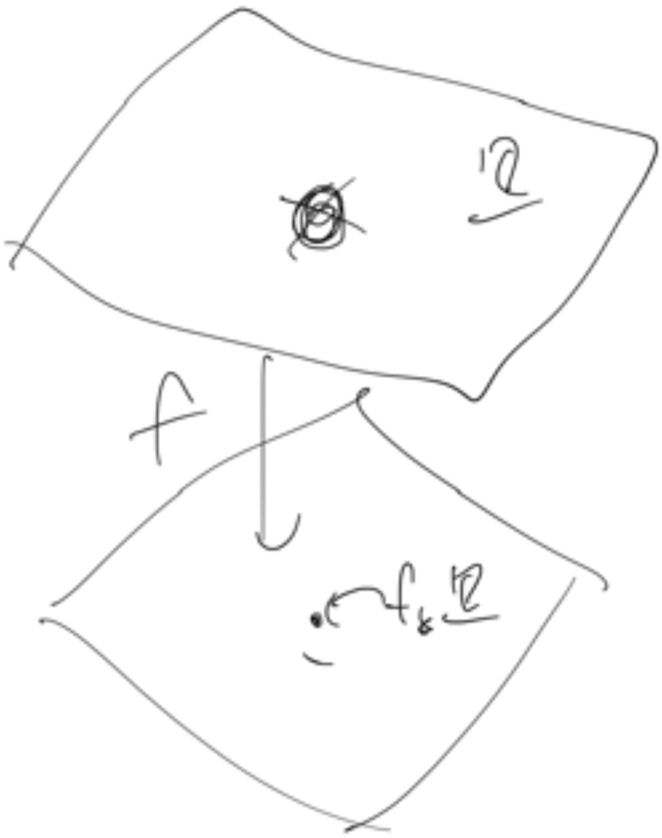
<u>Thm</u> $f: X \to Y$ proper $\Rightarrow$

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & \ulcorner & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

$$g^* f_* \simeq f'_* g'^*$$

In particular: stalks of f_* = coh of

fibers.



$$(f_* \underline{\mathbb{Z}})_0 = \varinjlim_{U \ni 0} (f_* \underline{\mathbb{Z}})(U)$$

$$= \varinjlim_{U \ni 0} \underline{\mathbb{Z}}(f^{-1}(U))$$

$$= \varinjlim_{U \ni 0} RI(U \setminus 0; \mathbb{Z})$$

$$= \underline{\mathbb{Z}}[-1] \oplus \underline{\mathbb{Z}}$$

So f_* is bad.

Replacement: $f_!$

designed so that

$f_! = f_*$

if f proper, &

$f_!$ always satisfies

proper base change

(& comm w/

colims)

$f_!$ = derived analog

of <u>sections</u>

<u>w/ proper support</u>

$f : X \to Y$

$\mathcal{F} \in Sh(X; D(R))$

$$(f_! \mathcal{F})(V) = \lim_{\substack{K \subset X \\ \downarrow \text{proper} \\ V \subset Y}} \mathrm{Fib}\left(\mathcal{F}(X) \to \mathcal{F}(X \setminus K)\right)$$

Has right adjoint $f^!$

$$f^! \mathcal{G}(U) = \mathrm{RHom}(f_! \mathbb{Z}_U, \mathcal{G})$$

Good for proving duality

e.g.:

<u>Thm (Poincaré Verdier)</u>: M top manifold. Then for $M \xrightarrow{p} *$ we have

$$f^! V \simeq (f^* V) \otimes f^! \mathbb{Z}$$

$\mathbb{Z}$ if M orientable //

$\omega[d]$,

$d = \dim M$.

Pf: For a open subset

$U \overset{j}{\hookrightarrow} M$,

$j^* = j^!$ $\Rightarrow$

$p^!\mathbb{Q}$ localizes on M.

We check on $\mathbb{R}^d$

that $p^!\mathbb{Q} = \mathbb{Q}[d]$,

then you're done.

$R\Gamma_c(\mathbb{R}^d) = \mathbb{Q}[-d]$.

Cor:

$p_! \mathbb{Q}$

$R\Gamma_c(M,\mathbb{Q}) =$

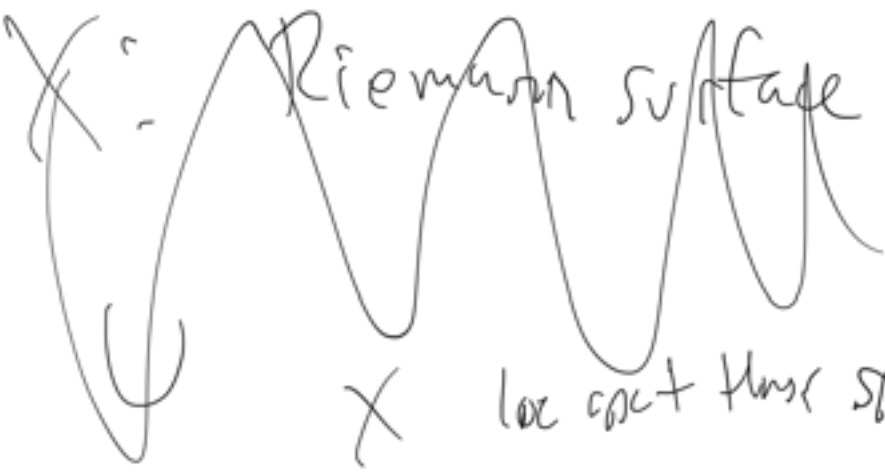
$R\mathrm{Hom}(p_!\mathbb{Q}, \mathbb{Q})$

$= R\mathrm{Hom}(\mathbb{Q}, p^! \mathbb{Q})$

$= p_* p^! \mathbb{Q}$

$= p_* p^* \mathbb{Q}[d]$ (or)

$= R\Gamma(M; or)$



X loc cpct Haus space

Lurie: $\mathrm{Sh}(X; D(\mathbb{Q}))$

↓

$\mathrm{Sh}($site of cpct subsets of X$; D(\mathbb{Q}))$

↑ topology of finite unions

ess. image is those sheaves of

sheaves $\mathcal{G}$

s.t.

$$\mathcal{G}(K) = \varinjlim_{K \Subset K'} \mathcal{G}(K')$$

$$\mathcal{F} \in Sh(X)$$

$$\downarrow$$

$$\mathcal{F}(K) = \varinjlim_{U \supseteq K} \mathcal{F}(U)$$

$$= \Gamma(i^*_{K \subset X} \mathcal{F})$$

~~$\exists$~~ $\mathcal{G} \in Sh(\text{compacts})$

$$\Rightarrow \mathcal{F}(U) = \varprojlim_{K \subset U} \mathcal{G}(K)$$

X Riemann surface.

$U \subset X \rightsquigarrow \mathcal{O}(U)$:

ring of holomorph

w/ us, viewed as a condensed (even liquid) $\mathbb{C}$-module.

$$S \longrightarrow \mathcal{O}(U)$$

= cts maps

$$\begin{array}{ccc} S \times U & \longrightarrow & \mathbb{C} \\ \uparrow & \nearrow_{\text{holomorphic.}} & \\ \{s\} \times U & & \end{array}$$

s.t.

$$\mathcal{O}(U) \subset C^\infty(U) \xrightarrow{\bar{\partial}}$$

$\mathcal{O}(\overline{D} \subset X)$ = what we had before, overconvergent holo fns on $\overline{D}$.

Consider the derived category of

Consider sheaves of derived liquid $\mathbb{C}$-modules on X w/ action of $\mathcal{O}(-)$.

<u>Def'n:</u> Such an $\mathcal{O}$-module sheaf $\mathcal{M}$ is <u>quasi-coherent</u> if $\forall$

$$\overline{D} \subset \overline{D}'$$

inclusions of closed disks,

$$\mathcal{O}(\overline{D}) \overset{L}{\underset{\mathcal{O}(\overline{D}')}{\otimes}} \mathcal{M}(\overline{D}')$$

$$\downarrow \simeq$$

$\mathcal{M}[\bar{D}]$

"closed disks $\leftrightarrow$ affines"

We can define a functor

$$f : X \to Y$$

(only interested in 2 cases:

$$U \overset{j}{\hookrightarrow} X, \quad j \text{ open immersion}$$

$$X \xrightarrow{p} *$$)

functors $f_!$ exactly as in the $Sh(X; D(\mathbb{Z}))$ case, and (claim) they such $\mathcal{O}$-modules to $\mathcal{O}$-modules, & preserve quasi-coherence.

$$(f_! \mathcal{F})(S) = f_! \mathcal{F}(S)$$

S exh loc.

$f_!$ comm w/ colim, has a right adjoint $f^!$. Key:

Prop:

Thm: $f^!(V) \simeq f^*(V) \otimes f^!(\mathbb{C})$, $f^!(\mathbb{C}) \simeq \Omega^1[1]$.

Key: Understand what

$\mathrm{QCoh}(D)$

$\uparrow$

open disk

looks like.

<u>Claim:</u> $D \subset \bar{D}$

$\mathrm{QCoh}(D) \simeq$ Full subcategory of $G(\bar{D})$-modules in liquid $\mathbb{C}$-modules.

M s.t. $M \otimes_{G(\bar{D})} G(\bar{D} \setminus D) = 0$

Functor: $\leftarrow$

$E \subset D$, $\longleftarrow\!\!\!\!\mid$ M

$\mathcal{F}(E) = M \otimes G(E)$

$\mathcal{O}(\bar{D})$

$\longrightarrow$

$\mathcal{F} \mapsto j_*(j^! \mathcal{F}(\bar{D}))$

$\parallel$

$\mathcal{F}(*D)$

$\mathcal{F}(*D)$

$\mathrm{Fib}\left(\mathcal{F}(D) \to \mathcal{F}(\bar{D}) \otimes_{\mathcal{O}(\bar{D})} \mathcal{O}(\bar{D} \setminus D)\right)$

Rk: $\mathcal{O}(\bar{D} \setminus D)$

is an idempotent

$\mathcal{O}(\bar{D})$-algebra.

Eg $\mathrm{QCoh}(D)$ is

gotten from $\mathcal{O}(\bar{D})$

by killing an

idempotent algebra.

$\Rightarrow$ descriptions of

$$p_*, \; p^*, \; p_!, \; p^!$$

~~$p^* V = \underline{RHom}_{G(\bar D)}\big(\mathrm{Fib}(G(\bar D) \to G(\partial \bar D)), \; V \otimes G(\bar D)\big)$~~

$$p^! V = \underline{RHom}_{G(\bar D)}\big(\mathrm{Fib}(G(\bar D) \to G(\partial \bar D)), \; V \otimes G(\bar D)\big)$$

reduced to calculating

$$\mathrm{Fib}\big(G(\bar D) \xrightarrow{c} G(\partial \bar D)\big)$$

= ~~Koszul of~~ 1-forms on "compactly closed disk in P"

Conclusion:

$$p^! \simeq p^* \otimes \Omega^1[1]$$

Cor: Note that $\prod$ commutes w/ colimits, i.e. right adjoint of $p_!$ commutes w/ all colimits.

$\Rightarrow$ ~~stacky~~ $p_!$ sends cpct objects to cpct objects.

vector bundle is dualizable, hence is compact $\Leftarrow$ $\mathcal{O}(1$

is compact (⟹)

P_∞ commutes w/ all colims.

⟹ if X cpct, get that

$\mathbb{R}\Gamma(X; V) = P_1 V = P_\infty V$

is a compact object in Rep.

OTOH, this $\mathbb{R}\Gamma$ is a ~~[struck out]~~ finite limit of $G(\mathbb{F})^{(\oplus n)}$'s

⟹ "nuclear" liquid module.

Thus we use

nuclear + cpct =

Perfect

⇒ finite dim'l,

finiteness of

$R\Gamma(X;V)$.

⇒ Serre duality

just as in

Verdier setting.

Recall: $M^{\in D(\mathrm{Nuc}_p)}$ Nuclear

⟺ $\forall$ S extr.

$(C(S,\mathbb{C}) \overset{L}{\otimes} M)(*) \to M(S)$

$\cong$

Banach spaces are

not flat wrt

liquid $\otimes$.

We don't know whether $\otimes$ closed under $\pi_i(-)$.

$\Leftarrow$ gen under colims by "basic nuclear"

$$\varinjlim \left(M_0 \to M_1 \to \cdots \right)$$

cpt proj.; tr. class

special case: $\to$ are inclusions.

$\Rightarrow$ DNF.

Recall:

dualizable $\Leftrightarrow$ nuclear + compact.

||

for $\forall$ colims $\left(M^{(-f)}, M \right)$