# Rigid $C^*$-tensor categories, Quantum Groups, + Applications

**Goals**: Consider CQGs from a categorical perspective.

**Main idea**: $G \longrightarrow \mathcal{C} = \mathrm{Rep}(G)$ rep. category

$$\underbrace{\text{rigid } C^*\text{-tensor category}}_{\text{Study this as an abstract structure.}}$$

→ Connections to subfactors, topological quantum field theory, topological phases of matter, ---

→ $\mathrm{CQG} \rightsquigarrow$ (analysis on $C^*$-t categories, subfactors, etc... categorical ideas (monoidal equivalence, fibre functors))

new results $\leftsquigarrow$ about CQG (eg. Vaes)

→ Ideas of categorical nature appear in quantum information problems

**Lecture 1**: Given a compact quantum group

$$G = (\mathcal{O}(G), \Delta, S, \mathcal{E}) \longrightarrow \underbrace{\text{study it's unitary representations of } G}_{\text{encode in category. Rep } G}$$

$$\mathcal{C} = \mathrm{Rep}\, G$$

$$\begin{cases} Ob(\mathcal{C}) = \text{unitary f.d. representations}^{\,u} \text{ of } G. \\ \text{Morphisms} = Mor(\mathcal{C}) = \{Mor(u,v)\}_{u,v \in Ob(\mathcal{C})} \end{cases}$$

__Additional structure__ : (1) $\forall$ $u, v \in \mathcal{C}$, $Mor(u,v)$ is a __Banach space__, and composition $Mor(u,v) \times M(w,u) \longrightarrow Mor(w,v)$

$$S \times T \longmapsto ST$$

is contractive $\|ST\| \leq \|S\| \cdot \|T\|$

(2) __Involutive structure__: i.e. a $\overset{\text{antilinear}}{\text{contravariant}}$ functor

$* : \mathcal{C} \longrightarrow \mathcal{C}$, identity on $Ob(\mathcal{C})$,

$* : Mor(u,v) \longrightarrow Mor(v,u)$, $T \longmapsto T^*$

Satisfying

(a) $T^{**} = T$

(b) $\|T^*T\| = \|T\|$ ($\Longrightarrow$ $End(u) = Mor(u,u) =$ a $C^*$-algebra)

(c) $\forall$ $T \in Mor(u,v)$, $T^*T \geq 0$ in $End(u)$.

$\rightsquigarrow$ The above axioms describe a __$C^*$- Category__.

In the case $\mathcal{C} = \mathrm{Rep}(G)$, we also have

- ∃ of **direct sums**: given $u, v \in \mathcal{C}$,

  ∃ $w = "u \oplus v"$.

  ⇝ can find isometries $S \in \mathrm{Mor}(u, w)$
  $T \in \mathrm{Mor}(v, w)$

  st $id_W = SS^* + TT^*$

- ∃ **subobjects**: Given $p \in \mathrm{Mor}(u,u)$ $p^2 = p = p^*$,

  ∃ $v \in \mathcal{C}$, and an isometry $T \in \mathrm{Mor}(v, u)$ st.

  $$VV^* = p$$

  → A general $\mathcal{C}$ can always be completed to have this property.

**Tensor Structure**: In $\mathrm{Rep}(G)$, can form the tensor product $u \otimes v := u_{13} u_{23}$

**bilinear bifunctor** $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ $(u, v) \longmapsto u \otimes v$
$(S, T) \longmapsto S \otimes T$

with properties: **strictness** $u \otimes (v \otimes w) = (u \otimes v) \otimes w$ ∀ $u, v, w$

(in general, can allow associator isomorphisms
$\alpha_{u,v,w} : (u \otimes v) \otimes w \xrightarrow{\sim} u \otimes (v \otimes w)$ + pentagon eq'n)

Also have a distinguished trivial object

$\mathbb{1} \in \mathcal{C}$, st. $\mathbb{1} \otimes u \cong u \cong u \otimes \mathbb{1}$ $\forall u$

Moreover, $(\mathcal{C}, \otimes)$ is called a (Strict) $C^*$-tensor category.

(typically assume here $\mathbb{1} \in \mathcal{C}$ is simple

i.e. $\mathrm{Mor}(\mathbb{1}, \mathbb{1}) = \mathbb{C}1$

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Another example: (to be explained by Vaes)

$N \subset M$ $\mathrm{II}_1$-factors $\rightsquigarrow \mathcal{C} = {}_N\mathrm{Mod}_N$, $\mathcal{C} = {}_M\mathrm{Mod}_M$

$\mathcal{C} \in {}_N\mathrm{Mod}_N$ gen by $\left({}_N L^2(M_k)_N\right)_k$

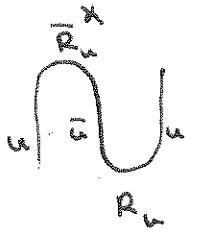
Morphisms = N-N-bimodule maps.

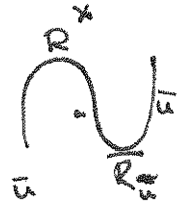
# Conjugate Objects

$\mathcal{C}$ - strict $C^*$-tensor category. Fix $u \in \mathcal{C}$.

$\bar{u} \in \mathcal{C}$ is called conjugate to $u$ if

∃ morphisms $R_u \in \mathrm{Mor}(\mathbb{1}, \bar{u}\otimes u)$, $\bar{R}_u \in \mathrm{Mor}(\mathbb{1}, u\otimes\bar{u})$ satisfying the zig-zag relations:

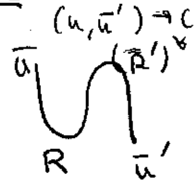
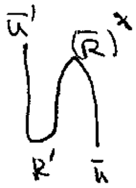
$1_u =$  $= (\bar{R}^*\otimes\iota)(\iota\otimes R)$

$1_{\bar{u}} =$  $= (R^*\otimes\iota)(\iota\otimes\bar{R})$

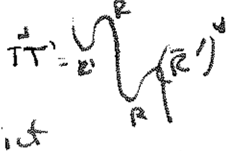
$(R,\bar{R})$ are a solution to the conjugate equations for $(u,\bar{u})$.

**Fact**: $\bar{u}$ is unique, if it exists. (and $R' = (T'\otimes\iota)R$, $\bar{R}' = (\iota\otimes T^*)\bar{R}$

( **Idea** $(u,\bar{u}) \to (R,\bar{R})$ $T\in\mathrm{Mor}(\bar{u}',\bar{u})$
$(u,\bar{u}') \to (R',\bar{R})$

$T =$  , $T^{-1} =$  )

$\mathcal{C}$ is called rigid if $\forall u\in\mathcal{C}$ $\exists\, \bar{u}\in\mathcal{C}$ conjugate to $u$.

$TT' =$  $= \mathrm{id}$

**Aside**: We often use diagrammatic calculus to depict $\mathrm{Mor}(\mathcal{C})$.

$T\in\mathrm{Mor}(u,v) \longleftrightarrow$ 

$T\in\mathrm{Mor}(u\otimes v, w)$

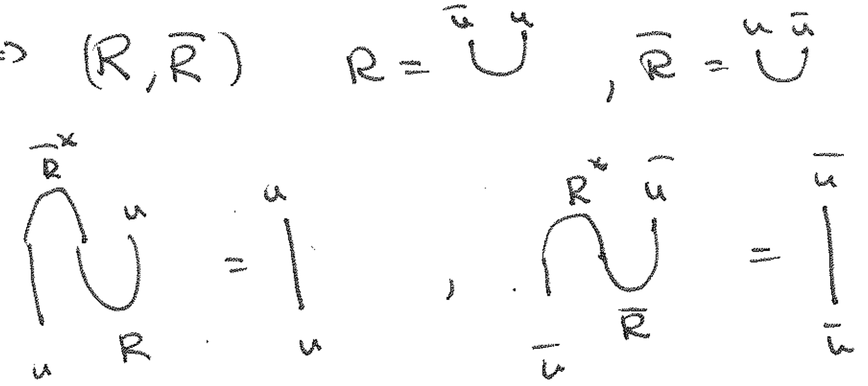
↕

$S\otimes T \longleftrightarrow$ 

# Lecture II

$\mathcal{C}$ - $C^*$-tensor category (strict)

$(u, \bar{u})$ conjugate $\Leftrightarrow$ $(R, \bar{R})$



St.

Say $\mathcal{C}$ is <u>rigid</u> if every object has a <u>conjugate</u>

**Ex:** $\mathrm{Hilb}_f$ → rigid

$(H, \bar{H})$ conjugate,

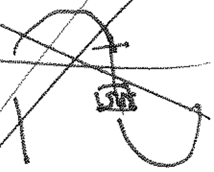
put $r = \sum_i \bar{e}_i \otimes e_i$, $\bar{r} = \sum_i e_i \otimes \bar{e}_i$

check: $(r, \bar{r})$ solve conjugate eq'ns.

~~But, also $R = (\iota \otimes T) r$, $\bar{R} = (T^{*-1} \otimes \iota) \bar{r}$ $\forall\, T \in GL(H)$~~

~~$\bar{R} =$ ( [illegible] ) [illegible]~~

~~work~~



But also, $\forall\ T \in GL(H)$,

get $$R = (\iota \otimes T) r = (j(T) \otimes \iota) r$$
$$\bar{R} = (T^{-1} \otimes \iota) \bar{r} = (\iota \otimes j(T)^{-1}) \bar{r}$$

another pair of solutions.

$$\left( \underline{\text{Recall}}: \quad \begin{aligned} j : B(H) &\longrightarrow B(\bar{H}) \\ j(T)\bar{\xi} &= \overline{T^{*}\xi} \end{aligned} \right) \quad \text{*-anti-isomorphism}.$$

## For a compact quantum group $G$:

Given $u \in \mathrm{Rep}(G)$, $\quad u \in B(H) \otimes \mathcal{O}(G)$

put $u^c = (j \otimes \iota)(u^{-1}) \rightsquigarrow$ a f.d. rep of $G$ on $\bar{H}$

but may not be unitary!

**Ex:** $G = SU_q(2)$ : $\mathcal{O}(G) = \mathrm{Alg}^*\left(u_{ij}\ 1 \le i,j \le 2 \;\middle|\; u = \begin{bmatrix} \alpha & q\gamma^* \\ \gamma & \alpha^* \end{bmatrix} \text{ unitary}\right)$

$$\Delta(u_{ij}) = \sum_{k=1}^{2} u_{ik} \otimes u_{kj}$$

$u$ - rep (unitary), irred

but $u^c = \begin{bmatrix} \alpha^* & -q\gamma \\ \gamma^* & \alpha \end{bmatrix}$ is <u>NOT</u> unitary

**But**, letting $\rho = \begin{bmatrix} |q|^{-1} & 0 \\ 0 & |q| \end{bmatrix}$, get $(j(\rho)^{1/2} \otimes \iota)\, u^c\, (j(\rho)^{-1/2} \otimes \iota) \in B(\bar{H}) \otimes \mathcal{O}(SU_q(2))$
unitary (check!)

## For general $G$

for $u$ irreducible,

Take $Q_r = (\iota\otimes h)(u^{c*}u^c) \in B(\overline{H_u})$

$Q_\ell = (\iota\otimes h)(u^c u^{c*}) \in$

$$Q_\ell Q_r = \lambda 1 > 0$$

$\Rightarrow$ Let $\rho_u = \frac{j(Q_r)}{\sqrt{\lambda}} \in B(H_u)$

Then $Tr(\rho_u) = Tr(\rho_u^{-1}) \; \left(= \dim_q(u)\right)$

AND $(j(\rho_u)^{1/2}\otimes\iota)\, u^c\, (j(\rho_u)^{-1/2}\otimes 1) =: \bar{u}$

unitary (so $j(\rho_u)^{1/2} \in Mor(u^c, \bar{u})$

also $\underset{\rho_u^{-1/2}}{\overset{\|}{j(\rho_{\bar u})^{1/2}}} \in Mor(\bar{u}^c, \underset{u}{\overset{\|}{\bar{\bar{u}}}})$

Claim $(u, \bar{u})$ conjugate

indeed $\bar{r} \in Mor(1\!\!1, u\otimes u^c)$

$j(\rho_u)^{1/2} \in Mor(u^c, \bar{u})$,

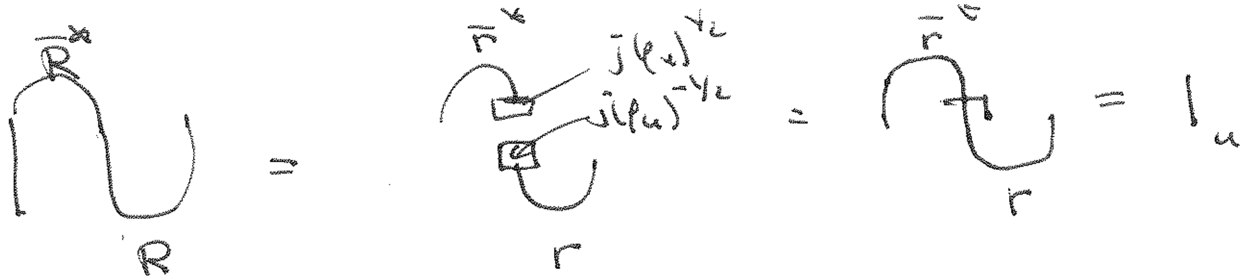
get $\bar{R}_u := (\iota\otimes j(\rho_u)^{1/2})\bar{r} \in Mor(1\!\!1, u\otimes\bar{u})$

repeating with $\bar{u}$, get

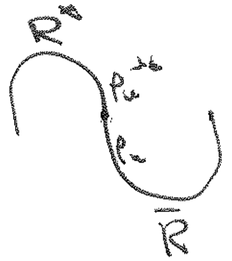
$r \in Mor(1\!\!1, \bar{u}\otimes\bar{u}^c)$

$R_u : (\iota\otimes j(\rho_{\bar u})^{1/2})\, r = (\iota\otimes \rho_u^{-1/2})\, r \in Mor(1\!\!1, \bar{u}\otimes u)$

Then check



Same for



# Consequences of rigidity

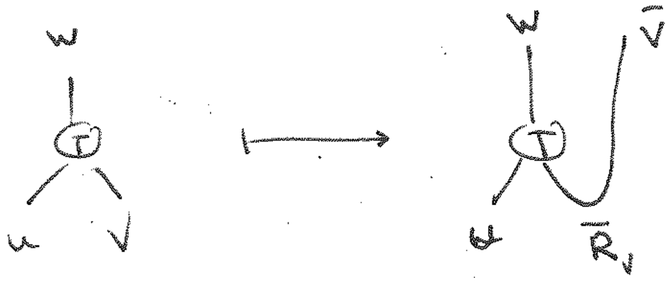
## Frobenius Reciprocity

$\mathcal{C}$ - rigid $C^*$ TC

$\forall\, u, v, w \in \mathcal{C}$, have linear isomorphisms

$$\mathrm{Mor}(u \otimes v, w) \cong \mathrm{Mor}(v, \bar{u} \otimes w)$$

$$\mathrm{Mor}(u \otimes v, w) \cong \mathrm{Mor}(u, w \otimes \bar{v})$$



**Corollary**: $\bar{u}$ is simple $\Leftrightarrow$ u is simple

Pf: $E(\bar{u}) = \mathrm{End}(\bar{u}, \bar{u}) \cong \mathrm{End}(u, u)$

# Categorical Trace:

If $u \in \mathcal{C}$ simple, $\bar{u}$ conjugate. Then

$(R, \bar{R})$ any other solution for $(u, \bar{u})$ is of the form

$$R' = \lambda R \qquad \Rightarrow \text{ choose } R, \bar{R} \text{ st. } \|R\| = \|\bar{R}\|.$$
$$\bar{R}' = \bar{\lambda}^{-1} \bar{R}$$

dimension $\quad d_i(u) = \|R\| \cdot \|\bar{R}\| = \|R\|^2 = \|\bar{R}\|^2 = d_i(\bar{u})$

for general $\quad u \cong \oplus_k u_k \qquad d_i(u) = \sum_k d_i(u_k) = \sum_k \|R_k\|^2$

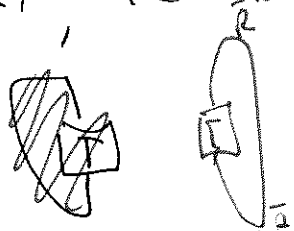
Build a standard solution $(u, \bar{u})(R, \bar{R})$

$$R_u = \sum_k (\bar{W}_k \otimes W_k) R_k$$

$$\bar{R}_u = \sum_k (W_k \otimes \bar{W}_k) \bar{R}_k$$

$$\Longrightarrow \quad d_i(u) = \|R\|^2 = \|\bar{R}\|^2 = \|R\| \cdot \|\bar{R}\|$$

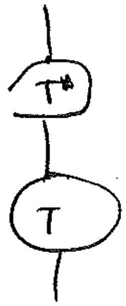
for this choice of standard $(R, \bar{R})$, let

$\varphi_L(T) =$  , $\varphi_R(T) =$ 

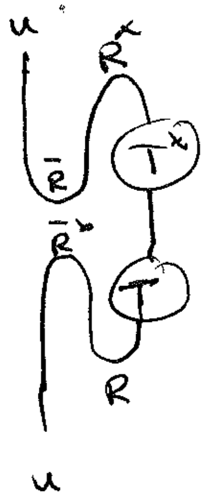
These are traces, $\varphi_L = \varphi_R$, $\varphi_L(T^*T)\|R\|^2 \mathbb{1} \geq T^* T$ $\Rightarrow$ End(u) is f.d.

For End(u) - f.d., $T \in \mathrm{End}(u)$

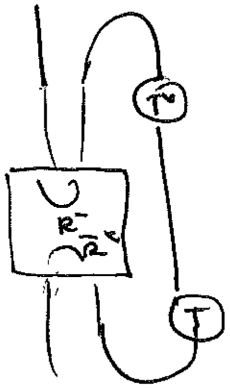
Write $T^*T =$



$=$



$=$



$$\leq \|\bar{R}\|^2 1_u \varphi(T^*T)$$

$\rightsquigarrow$ get $\varphi \geq 0$ on End(u) st.

$$C\varphi(T)1 \geq T \quad \forall \; T \in \mathrm{End}(u)_+$$

# Lecture III:

Recall: $d_i(u) = \dim(u) = \|R\|\cdot\|\bar{R}\|$, $(R,\bar{R})$ solve conj. eq'ns for $(u,\bar{u})$

Intrinsic dimension

$$d_i(u) := \sum_k d_i(u_k) \quad \text{if} \quad u \simeq \bigoplus_k u_k$$

For Rep G: $\dim(u) = \dim_q(u) = \mathrm{Tr}(\rho_u) = \mathrm{Tr}(\rho_u^{-1})$

(if $u$ reducible $\Rightarrow \rho_u = \oplus_k \rho_{u_k}$)

## Categorical Trace:

Given $u \in \mathrm{Irr}(\mathcal{C})$, fix $(R,\bar{R})$ for $(u,\bar{u})$

St. $\|R\| = \|\bar{R}\| = \sqrt{d_i(u)}$

Given $u \in \mathcal{C}$, $u \simeq \bigoplus_k u_k$, $u_k \in \mathrm{Irr}(\mathcal{C})$

Pick $\bar{R}_k, R_k$ as above for each $k$, fix $\forall k$

isometric $w_k \in \mathrm{Mor}(u_k, u)$

$\bar{w}_k \in \mathrm{Mor}(\bar{u}_k, \bar{u})$ ).

Put
$$\left.\begin{aligned} R &= \sum_k \bar{w}_k \otimes w_k \circ R_k \\ \bar{R} &= \sum_k w_k \otimes \bar{w}_k \circ \bar{R}_k \end{aligned}\right)$$

check: solve conj eq'ns for $(u,\bar{u})$

$(R, \bar{R})$ – "std solutions" to CE's

$\rightsquigarrow$ define

$\varphi_L(T) =$

, $\varphi_R(T) =$

Fact: $\varphi_L = \varphi_R = \mathrm{Tr}_u$ is a trace. (Categorical Trace)
(independent of std solution choice)

Have $\mathrm{Tr}_{u \oplus v}(S \oplus T) = \mathrm{Tr}_u(S) + \mathrm{Tr}(T)$

$\mathrm{Tr}_{u \otimes v}(S \otimes T) = \mathrm{Tr}_u(S)\mathrm{Tr}(T)$

$\Rightarrow \quad d_i(u) = \mathrm{Tr}_u(1), \quad d_i(u \otimes v) = d_i(u) d_i(v)$

Main idea: $U \simeq \bigoplus_\alpha (1_{n(\alpha)} \otimes u_\alpha)$, $u_\alpha$ irrep, inequiv.

$$\Rightarrow \quad R = \sum_\alpha \left( \sum_{j=1}^{n(\alpha)} \overline{w_j^{(\alpha)}} \otimes w_j^{(\alpha)} \circ R_\alpha \right)$$

$$\bar{R} = \sum_\alpha \left( \sum_j w_j^{(\alpha)} \otimes \overline{w_j^{(\alpha)}} \circ \bar{R}_\alpha \right)$$

$$\Rightarrow \varphi_L(T) = \sum_\alpha \sum_{j,i} \left( w_i^{(\alpha)*} T w_j^{(\alpha)} \right) 1_{u_\alpha}$$

$$= \sum_\alpha \mathrm{Tr}\left[ \left( w_i^{(\alpha)*} T w_j^{(\alpha)} \right) \right] d_i(u_\alpha)$$

$$\varphi_L(T) = \sum_\alpha \mathrm{Tr}(T^{(\alpha)}) d_i(u_\alpha)$$

# Tensor Functors and
# Woronowicz - Tannaka-Krein Reconstruction

$\mathcal{C}, \mathcal{C}'$ - $C^*$-tensor categories (strict, rigid)

A unitary tensor functor is a functor

$F : \mathcal{C} \to \mathcal{C}'$ satisfying

$F(\mathbb{1}_{\mathcal{C}}) \simeq \mathbb{1}_{\mathcal{C}'}$

together with unitary isomorphisms

$F_2 : F(u) \otimes F(v) \longrightarrow F(u \otimes v) \quad \forall\, u, v \in \mathcal{C}$

satisfying $\otimes$-compatibility conditions:

$$\begin{tikzcd}
F(u) \otimes F(v) \otimes F(w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & F(u \otimes v) \otimes F(w) \arrow[r] & F((u \otimes v) \otimes w) \arrow[d] \\
F(u) \otimes F(v \otimes w) \arrow[rr] & & F(u \otimes (v \otimes w))
\end{tikzcd}$$

$$\begin{tikzcd}
F(\mathbb{1}) \otimes F(u) \arrow[r] \arrow[d, phantom, "\wr"] & F(\mathbb{1} \otimes u) \arrow[d, phantom, "\wr"] \\
\mathbb{1}' \otimes F(u) \arrow[r, phantom, "\simeq"] & F(u)
\end{tikzcd}$$

Same for other side

Say that $\mathcal{C}, \mathcal{C}'$ are unitarily monoidally equivalent if $\exists$ a unitary tensor functor

$$F: \mathcal{C} \longrightarrow \mathcal{C}' \quad \text{st.} \qquad \left( \mathcal{C} \xrightarrow[\sim]{F} \mathcal{C} \right)$$

(1) $F : \mathrm{Mor}(u,v) \longrightarrow \mathrm{Mor}(F(u), F(v))$ is an isomorphism

$\forall\ u, v \in \mathcal{C}$.

(2) If $\{u_\alpha\}_{\alpha \in A}$ is a complete set of inequiv. representatives of

$\mathrm{Irr}(\mathcal{C})$,

$\Longrightarrow \{F(u_\alpha)_{\alpha \in A}\}$ is a " " " for $\mathrm{Irr}(\mathcal{C}')$

Say two CQGs are monoidally equivalent $\iff \mathrm{Rep}G_1 \overset{F}{\sim} \mathrm{Rep}G_2$

Ex: Later $G_X, G_Y$ monoidally equivalent

$X, Y$ finite graphs with $F(A_X) = A_Y$

iff graph isomorphism game $\mathrm{Iso}(X,Y)$

has a winning qc-strategy.

Ex: $F \in GL_n(\mathbb{C})$, $F\bar{F} = 1$, $\rightsquigarrow O_F^+ \rightsquigarrow \mathcal{O}(O_F^+)$

$$\mathcal{O}(O_F^+) = \text{*-alg}\left(u_{ij} \;\middle|\; \begin{array}{l} u \text{ unitary} \\ F\bar{u}F^{-1} = u \end{array}\right)$$

Then $O_F^+ \overset{mon}{\sim} SU_{-q}(2)$, where $q + q^{-1} = \mathrm{Tr}(F^*F)$ $(0<q<1)$

**Def'n**: A fiber functor ~~F~~ on $\mathcal{C}$ (unitary)

is a unitary tensor functor

$$F : \mathcal{C} \longrightarrow Hilb_f$$

⊗ **Fact**: F is automatically faithful

(F is injective on morphisms)

non-zero on any object $u \in \mathcal{C}$

$F(u) \neq 0 \;\forall\; u \in \mathcal{C}$.

Sup $\delta(\eta)T_{\alpha,\beta} = T\eta \;\forall\; T \in Mor(\delta, \alpha\times\beta$

$\eta \in B(H_\delta)$

## Theorem: (Woronowicz - Tannaka - Krein)

$\mathcal{C}$ - rigid $C^*$-tensor categories,

$F : \mathcal{C} \to Hilb_f$ a unitary fiber functor.

$\Rightarrow \exists!$ a CQG $G = (\mathcal{O}(G), \Delta)$ and a unitary monoidal equivalence $E : \mathcal{C} \to Rep\, G$.

**Idea**: Given $G = (\mathcal{O}(G), \Delta)$, $Irr(Rep(G)) \sim \{V_\alpha\}_{\alpha\in A}$.

Define ~~$H_\alpha$~~

$$\mathcal{U}(G) = \left(\prod_\alpha B(H_\alpha) = \right) \mathcal{O}(G)^*$$

$$\eta_1\eta_2 = (\eta_1 \otimes \eta_2)\Delta(\cdot) \quad , \quad \eta^*(a) = \overline{\eta(S(a^*))}$$

$$\delta(\eta)(a\otimes b) = \eta(ab) \quad \big( (\delta\otimes\iota)\delta = (\iota\otimes\delta)\delta \quad , \quad \delta : \mathcal{U}(G) \to \underset{\overset{\parallel}{(\mathcal{O}(G)\otimes\mathcal{O}(G))^*}}{\mathcal{U}(G\times G)}$$

Put $\mathcal{U}(G) = \underline{\mathrm{Nat}}(F,F) = (\eta_u)$ $\eta_u : F(u) \longrightarrow F(u)$, s.t. $\forall\, T \in \mathrm{Mor}(u,v)$, $F(T)\eta_v = \eta_u F(T)$

all natural trans. $F \to F$

$$= \prod_{\alpha} B(F(u_\alpha))$$

$$\mathcal{U}(G\times G) = \mathrm{Nat}(F^{\otimes 2}, F^{\otimes 2}) \quad !$$
$$=$$
$$\prod_{\alpha,\beta} B(\cancel{F}F(u_\alpha)\otimes F(u_\beta))$$

$$\delta : \mathcal{U}(G) \longrightarrow \mathcal{U}(G\times G)$$

$$(\delta\eta)_{u,v} = F_2^{*}\eta_{u\otimes v}F_2$$

**Observe** $(\delta\eta)_{u,v}F(T) = F_2^{*}\eta_{u\otimes v}\underbrace{F_2F(T)}_{F(w)\to F(u\otimes v)}$

$T \in \mathrm{Mor}(w, u\otimes v)$

$$= F_2^{*}F_2F(T)\,\eta_{\cancel{u\otimes v}\,w}$$
$$= F(T)\eta_w$$

from this, get $(\iota\otimes\delta)\delta = (\delta\otimes\iota)\delta$, $\delta$ - $*$-homom

**Next:** $\mathcal{O}(G) :=$ finitely supported dual of $\mathcal{U}(G)$

$$\cong \bigoplus_{\alpha} B(F(u_\alpha))^*$$

$a, b \in \mathcal{O}(G)$,

→ $(ab)(\eta) = (a \otimes b)(\delta\eta)$ (multiplication coassoc.)

→ $1(\eta) = \eta_{\mathbb{1}}$ (unit of ~~$\mathcal{A}$~~)

→ $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$

$$\Delta(a)(\eta \otimes \eta') = a(\eta\eta')$$

→ co-unit $\varepsilon(a) = a(1_{\mathcal{U}(G)})$

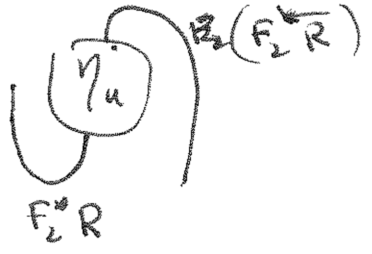
→ antipode: $S : \mathcal{O}(G) \to \mathcal{O}(G)^{op}$

→ involution $* : \mathcal{O}(G) \to \mathcal{O}(G)$

harder

Given $(u, \bar{u})$ ⇝ $(R, \bar{R})$ conjugates, $\eta \in \mathcal{U}(G)$ define

$\check{\eta} \in \mathcal{U}(G)$, $\check{\eta}_{\bar{u}} = (\eta_u)^\vee =$



**AND** then

$$S(a)(\eta) = a(\eta^{\vee})$$

$$a^*(\eta) = a(*\,\eta^{\vee *})$$

$\Rightarrow$ $(\mathcal{O}(G), S, \Delta, \varepsilon)$ Hopf $*$-alg.

**BUT** it's a CQG since it is the span of the unitary reps

$X^u \in B(F(u)) \otimes \mathcal{O}(G)$ ;

$$\boxed{(\iota \otimes \eta_u) X^u = \eta_u}$$

$$\left( \begin{array}{l} X^u \otimes X^v = (F_2^* \otimes \iota)\, X^{u \otimes v} (F_2 \otimes \iota) \\ \\ X^{u_\alpha} \text{ irreducible } \checkmark \end{array} \right.$$

$T \in \mathrm{Mor}(u, v)$

$\Rightarrow$ $F(T) \in \mathrm{Mor}(~~F(u)~~ X^u, X^v)$

~~(indeed~~, [illegible] ✓

$S \in \mathrm{Mor}(X^u, X^u)$

$\Rightarrow$ $\forall\, \eta$, $S\eta_u = S(\iota \otimes \eta_u) X^u = (\iota \otimes \eta_u)(S \otimes \iota) X^u$

$= (\iota \otimes \eta_u) X^u (S \otimes \iota)$

$= \eta_u S$

$\Rightarrow$ [illegible]

$Su_q(2) = O^{+}_{F_q} \}$ $\qquad F_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix} \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

$F\bar{F} =$ $\qquad F_q = \begin{pmatrix} 0 & |q|^{1/2} \\ -\mathrm{sgn}\,q\,|q|^{-1/2} & 0 \end{pmatrix} \begin{pmatrix} -\mathrm{sgn}\,q & 0 \\ \mathrm{sgn} & \mathrm{sgn}\,q \end{pmatrix}$

$$\boxed{F\bar{F} = -\mathrm{sgn}(q)}$$

---

$$R = R_1 + R_2$$

$$R = \sum_{k=1}^{2} (\bar{w}_k \otimes w_k) R_k$$

$$R^*(1 \otimes T) R = E$$

$$\sum_{k,l} R_l^* (\bar{w}_l^* \otimes w_l)(1 \otimes T)(\bar{W}_k \otimes W_k) R_l$$

$$= \sum_l R_l^* (1 \otimes W_l^* T W_k) R_l$$

$$= \sum_l \|R_l\|^2 T_{ll}$$

# Lecture IV

**Recall**: Two CQG's $G_1, G_2$ are *monoidally equivalent* $\left(G_1 \overset{\text{mon}}{\sim} G_2\right)$ if $\exists\ \mathrm{Rep}(G_1) \overset{\text{mon}}{\simeq} \mathrm{Rep}(G_2)$

↑ unitary monoidal equivalence

i.e have two unitary fiber functors

$$\begin{tikzcd}
 & \mathcal{C} = \mathrm{Rep}(G_1) \arrow[dl, "E"'] \arrow[dr, "F"] & \\
\mathrm{Hilb}_f & & \mathrm{Hilb}_f
\end{tikzcd}$$

**Ex:** $O^+_{F_1} \sim O^+_{F_2} \iff c_1 \mathrm{Tr}(F_1^* F_1) = c_2 \mathrm{Tr}(F_2^* F_2)$

$F_1 \in GL_{n_1}(\mathbb{C})$ $\quad$ $F_2 \in GL_{n_2}(\mathbb{C})$ $\quad$ Put $\delta = \mathrm{Tr}(F^* F)$

$F_1 \overline{F}_1 = c_1 1$ $\quad$ $F_2 \overline{F}_2 = c_2 1$

Why? One reason: $\mathrm{Rep}(O^+_F) \simeq TLJ(c, \delta)$

generated by $u$ - fund rep. $= \bar{u}$

$$\sum_{i=1}^n e_i \otimes F e_i$$

together with morphisms $\Big|_{u}^{u} = \mathrm{id}_{\mathbb{C}^n}$, $\underset{R_F}{\bigcup} = \underset{\overline{R}_F}{\bigcup}$

**Get:** $\mathrm{Mor}(u^{\otimes k}, u^{\otimes \ell}) = \mathrm{span}\{$  $\}_{p \in NC_2(k+\ell)}$

**Expect:** could define a ~~tensor functor~~ via

$$u_{F_1} \longmapsto u_{F_2}$$

can be extended to a unitary monoidal equivalence.

Indeed, it can! (Bichon - De Rijdt - Vaes)

**Actually more is true:** If $G$ ~~is~~ $O^+_F$, then

$G = O^+_{F'}$ for some $F'$.

So, $O^+_F$'s are closed under monoidal equivalence.

---

## Another perspective: on monoidal equivalence

Hopf bi-galois objects / linking algebras

Given $G_1, G_2$, how can we tell if $G_1 \overset{\mathrm{mon}}{\simeq} G_2$?
Especially if we don't know $\mathrm{Rep}(G_i)$?

L4-3

**Definition**: Let $G \overset{=}{\phantom{x}}{}^{(\mathcal{O}(G),\Delta)}$ be a CQG, $Z$ - unital $*$-algebra.

$A = \mathcal{O}(G)$. A left action of $G$ on $Z$ is

a unital $*$-homomorphism

$$\alpha : Z \to A \otimes Z = \mathcal{O}(G) \otimes Z$$

s.t. $$\begin{cases} (\iota \otimes \alpha)\alpha = (\Delta \otimes \iota)\alpha \\ (\varepsilon \otimes \iota)\alpha = \iota \end{cases}$$

A right action of $G$ on $Z$:

$$\beta : Z \to Z \otimes A \qquad \begin{cases} (\beta \otimes \iota)\beta = (\iota \otimes \Delta)\beta \\ (\iota \otimes \varepsilon)\beta = \iota \end{cases}$$

Write $G \overset{\alpha}{\curvearrowright} Z$, $Z \overset{\beta}{\curvearrowleft} G$.

A state $\omega : Z \to \mathbb{C}$ is called G-invariant

if $(\iota \otimes \omega)\alpha(z) = \omega(z)1 \quad \forall z \in Z$

$\left(\text{resp: } (\omega \otimes \iota)\beta(z) = \omega(z)1 \quad \forall z \in Z\right)$

We say two actions

$G_1 \overset{\alpha}{\curvearrowright} Z \xleftarrow{\beta} G_2$ commute if ~~$\alpha$~~ $\boxed{(\alpha\otimes\iota)\beta = (\iota\otimes\beta)\alpha}$ .

Call $G \overset{\alpha}{\curvearrowright} Z$ a left $\mathcal{O}(G)$ Galois extension

if the linear map

$$\kappa_\ell : Z\otimes Z \longrightarrow \mathcal{O}(G)\otimes Z \; ; \; \kappa_\ell(x\otimes y) = \alpha(x)(1\otimes y)$$

is bijective.

$Z \overset{\beta}{\curvearrowleft} G$ is a right $\mathcal{O}(G)$ Galois extension

if $\kappa_r : Z\otimes Z \longrightarrow Z\otimes\mathcal{O}(G) \; ; \; \kappa_r(x\otimes y) = (x\otimes 1)\beta(y)$

is bijective.

If $G_1 \curvearrowright Z \curvearrowleft G_2$ commute, and = left + right GE, $\Rightarrow$ say Z is a bigalois ext.

**Example**: $Z = \mathcal{O}(G)$ is a Bigalois extension, with an invariant state

**Theorem** (Bichon, Bichon - De Rijdt, Vaes)
(Ulbrich, Schauenburg - algebra context).

Let $G_1, G_2$ be compact quantum groups. Then

$$G_1 \sim G_2 \iff \exists \text{ a } \overset{\text{NON-ZERO!}}{\underline{\phantom{x}}}\ \mathcal{O}(G_1) - \mathcal{O}(G_2) \text{ - Bigalois extension}$$

$$G_1 \curvearrowright Z \curvearrowleft G_2$$

equipped with a bi-invariant state $\omega$.

**Sketch:** Sps. $G_1 \sim G_2$.

$$\begin{tikzcd} & \mathrm{Rep}(G_1) \arrow[dl, "E"'] \arrow[dr, "F"] & \\ \mathrm{Hilb}_f & & \mathrm{Hilb}_f \end{tikzcd}$$

$\iff$ two U.F. Functors

$\{u_\alpha\}_\alpha$ - reps of $\mathrm{Irr}(\mathrm{Rep}(G_1))$

Put $\mathrm{Nat}(E,F) \simeq \prod_\alpha B(E(u_\alpha), F(u_\alpha))$ (two sided analogue of $U(G)$)

$$\delta : \mathrm{Nat}(E,F) \to \mathrm{Nat}(E^{\otimes 2}, F^{\otimes 2}) = \prod_{\alpha,\beta} B(E(u_\alpha)\otimes E(u_\beta), F(u_\alpha)\otimes F(u_\beta))$$

$$\delta(\eta)_{u,v} = F_2^* \eta_{u\otimes v} E_2$$

i.e $\forall$ $T \in \mathrm{Mor}(w, u \otimes v)$, have

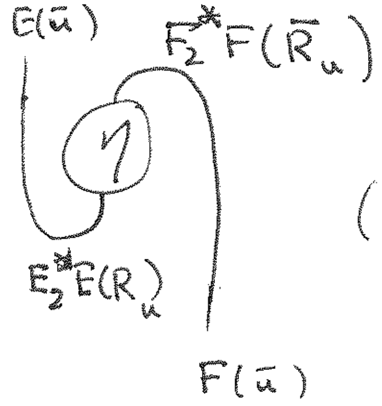
$$\boxed{\delta(\eta)_{u,v} E(T) = F(T)\eta_w}$$

Put $Z = \bigoplus_\alpha B(E(u_\alpha), F(u_\alpha))^* \subset \mathrm{Nat}(E,F)^*$

Put $x y \in Z$, $(xy)(\eta) = (x \otimes y)(\delta(\eta))$

$$x^*(\eta) = \overline{x(\eta^{\vee *})}$$

where $\check{\eta}_{\bar{u}} =$



$(R_u, \bar{R}_{\bar{u}})$ solve CEs for $u, \bar{u}$.

⤳ get a $*$-algebra $Z$, ~~together~~

For each $u \in \mathcal{C}$, put

$$X^u \in B(E(u), F(u)) \otimes Z, \quad (\iota \otimes \eta_u) Z = \eta_u.$$

<u>Facts</u>: - $X^u$ unitary $\forall u \in \mathcal{C}$, $Z$ = span coeff's of $X^u$ $\forall u$.

- $\forall$ $T \in \mathrm{Mor}(u, v)$, $(F(T) \otimes \iota) X^u = X^v (E(T) \otimes \iota)$

- $(F_2 \otimes 1) X^u_{13} X^v_{23} = (X^{u \otimes v})(E_2 \otimes 1)$

- $\omega : Z \to \mathbb{C}$ defined by

$$(\iota \otimes \omega)(X^u) = \delta_{u,1} 1$$ is a <u>faithful state</u>.

Finally, note that, we have actions of $\otimes$-alg's

$$\mathrm{Nat}(F) \curvearrowright \mathrm{Nat}(E,F) \curvearrowleft \mathrm{Nat}(E)$$

dualizing get commuting coa actions

$$\begin{tikzcd} \mathcal{O}(G_F) \arrow[r, bend left, "\alpha"] & Z & \mathcal{O}(G_E) \arrow[l, bend right, "\beta"'] \end{tikzcd}$$

$$((\iota \otimes \alpha) X^{u_\alpha} = U^\alpha_{12} X^{u_\alpha}_{13} \quad , \quad \alpha(z_{kl}) = \sum_{k'} u_{kk'} \otimes z_{k'l}$$

$$(\cancel{k} \otimes \beta)(X^u) = X^{u_\alpha}_{12} V^\alpha_{13} \qquad \beta(z_{kl}) = \sum_{l'} z_{kl'} \otimes V_{l'l}$$

$\rightsquigarrow$ can check bigalois condition directly.

$\rightsquigarrow$ $\omega$ is invariant.

<u>Converse</u>: Sps $\begin{tikzcd} G_1 \arrow[r, bend left, "\alpha"] & Z & G_2 \arrow[l, bend right, "\beta"'] \end{tikzcd}$, $\omega : Z \to \mathbb{C}$ invariant

is given. ~~Then~~ $\forall$

Given $\{u_\alpha\}_\alpha$ reps of $\mathrm{Irr}(G)$, $\mathcal{C} = \mathrm{Rep}(G_2)$

fix $\{H_\alpha\}$ corresponding

↳ viewed as left comodules

put

then, can define

$$F(u_\alpha) = H_\alpha \boxtimes Z = \{ x \in H_\alpha \otimes Z \mid (\delta_\alpha \otimes \iota)(x) = (\iota \otimes \delta_2)(x) \}$$

$\cong$ ↪ f.d. vector space; Hilbert space

$$F(T) = T \otimes \iota$$

defines a unitary fiber functor on $\mathrm{Rep}(G_2)$

Idea: use fact that $\delta_1, \delta_2$ commute, make

$$U^{F(u_\alpha)} \in B(H_{\varphi(x)}) \otimes$$

For $O^+_{F_1} \overset{\text{mon}}{\sim} O^+_{F_2}$,

$$\mathcal{Z}(F_1,F_2) = *\text{-alg}\left( W_{ij}\ ,\ \begin{matrix}1\le i\le n_1\\ 1\le j\le n_2\end{matrix} \;\middle|\; \overset{W \text{ unitary}}{F_1 W^c F_2^{-1} = W}\right)$$

$$\|$$

$\mathcal{O}(O^+_{F_1}) - \mathcal{O}(O^+_{F_2})$ - bicomodule algebra

$$(\iota\otimes\delta_1)(W) = (u^{F_1})_{12} W_{13}$$

$$(\iota\otimes\delta_2)(W) = W_{12} u^{F_2}_{13}$$

## Graphs and their q. automorphism groups, quantum graphs, monoidal equivalence:

$X$ - finite undirected graph on $n = |X|$ vertices, $A_X$ = adj matrix.

Saw: $G_X$ = quantum Aut. gp. of $X$ $= \mathrm{Qut}(X) = \mathrm{QAut}(X)$
$= \mathbb{G}_{aut}(X)$ - -

$$\mathcal{O}(G_X) := C(S_n^+) \Big/ \langle (A_X \otimes 1) u = u (A_X \otimes 1) \rangle$$

$$\Delta u_{ij} = \sum_k u_{ik} \otimes u_{kj}$$

→ Encodes the quantum symmetries of $X$.

→ classically $\mathrm{Aut}(X) = \{ P \in S_n \mid A_X P = P A_X )$

Ex: $X = K_n$ - complete graph on $n$ vertices

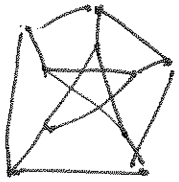
$$A_{K_n} = \begin{pmatrix} 1 & - & - & 1 \\ 1 & & & 1 \\ 1 & - & - & 1 \end{pmatrix}$$

$$G_X = S_n^+$$

Note: $G_X = G_{\bar{X}}$ b/c $A_{\bar{X}} = A_{K_n} - A_X$.

Questions: $\boxed{\text{Which } X \text{ have } Aut(X) \lneq G_X \text{ ?}}$

Hard! Ex: Petersen graph

$X =$  $\to Aut(X) = S_5$

Banica-Bichon 2007 - Classified all $G_X$ for $|X| < 11$, vertex transitive, except the Petersen graph

(Schmidt '18): $G_X = S_5$

?∃? $X$ st. $Aut(X) = \{e\}$, but $G_X \neq \{e\}$ ?

$\boxed{\text{Given } X, \text{ can we study } Rep(G_X)}$ ?

Not really! Understanding $G_X$ is hard!
$Rep(G_X)$ even harder!

$\boxed{\text{Given } X, Y : \text{Can we detect whether or not } G_X \simeq G_Y}$ ?

$\boxed{\text{Find } X \not\simeq Y \text{ s. that } G_X \overset{mon}{\simeq} G_Y \text{ ?}}$ ($X$, $Y$ interesting)

About $Rep(G_X)$: $Rep(S_n^+) = NC$

$$Mor_{S_n^+}(u^{\otimes k}, u^{\otimes \ell}) = span\{T_p : p \in NC(k,\ell)\}$$

For $G_X$: $\mathcal{O}(G_X) = \mathcal{O}(S_n^+) / \langle A_X u = u A_X \rangle$ $\iff$ Enlarging $Rep(S_n^+)$ to get $Rep(G_X)$ by adding $A_X \in Mor(u,u)$.

## Today's Problems:

(1) Given $X$, can we "describe" all CQG $\mathbb{G}$ st. $G_X \sim \mathbb{G}$?

(2) Is $\mathbb{G}$ always isomorphic to $G_Y$, for some $Y$?

For (2): NO! $K_n = X$; $S_n^+ = G_X$, then by [De Rijdt - Vander Vennet],

$$G_X \overset{mon}{\sim} SO_q(3), \quad q+q^{-1} = \sqrt{n}.$$

But, the answer to (1), (2) is yes if we enlarge our notion of graph $X$.

Def'n: A finite ~~qu~~ measured quantum set is

$$X = (\mathcal{O}(\cancel{X}), \psi_X)$$

$\mathcal{O}(X)$ - fd $C^*$-algebra (with maps $m, \eta$)

$\psi_X : \mathcal{O}(X) \to \mathbb{C}$ is a state

$\delta > 0$, $\psi_X$ is a $\delta$-form if $mm^* = \delta^2 1 : \mathcal{O}(X) \to \mathcal{O}(X)$

$\longrightarrow$ These are the "natural" states on $\mathcal{O}(X)$.

Ex $\mathcal{O}(X) = \mathbb{C}^n \rightsquigarrow \psi_X$ = uniform measure

is the unique δ-form

$\bar{w}$ $\delta = \sqrt{n}$

$\mathcal{O}(X) = M_n(\mathbb{C})$ $\exists!$ tracial δ-form $\bar{w}$ $\delta = n$.

Every state $\psi_X$ on $M_n(\mathbb{C})$ = δ-form with $\delta^2 = Tr(Q^{-1})$, if $\psi_X = Tr(Q\cdot)$

$\mathcal{O}(X)$ generally admits a unique tracial δ-form with $\delta = \sqrt{\dim \mathcal{O}(X)}$

⊛ Always consider $\psi_X$ a δ-form *

---

A quantum graph is measured q-set $(\mathcal{O}(X), \psi_X, A_X)$

(Musto-Ruetter-Verdon)

abelian δ-form adjacency matrix

with $A_X : L^2(X) \to L^2(X)$ linear map

$\quad\quad\; = [a^X_{ij}]$

st.

(1) $a^X_{ij} = {a^X_{ij}}^2 \longleftrightarrow m(A\otimes A)m^* = \delta^2 A$

(2) $a^X_{ij} = a^X_{ji} \longleftrightarrow (\iota\otimes\eta^* m)(1\otimes A\otimes 1)(m^*\eta\otimes\iota) = A_X$

(3) $a^X_{ii} = 1 \longleftrightarrow m(A\otimes\iota)m^* = \delta^2\iota$

Examples? (Non-classical)?

$A_X = id$ ✓, $A_X = \delta^2\psi_X(\cdot)1$ ✓ others??

**Rmk:** Another perspective:

A graph $X \equiv$ a symmetric, reflexive, relation

$$R_X \subset X \times X$$

$$(i,j) \in R_X \iff i \sim j \iff a_{ij} = 1$$

**Weaver** A quantum graph $X$, $\equiv$ a symmetric, reflexive quantum relation on a fd $C^*$-algebra

$$\underbrace{(\;). \; B \le B(H)}$$

$\equiv$ a $B'$-$B'$-Bimodule $E \subset B(H)$

$$\underbrace{\text{s.t. } B' \subset E = E^*.}_{\text{includes all operator systems! } E \subseteq M_n}$$

**Theorem:** (Musto-Reutter Verdon)

Let $X = (\mathcal{O}(X), \psi_X)$ be a q-set with $\psi_X$-tracial δ-form.

Then $\exists$ bijection between q. adj. matrices

$$A_X : L^2(X) \to L^2(X)$$

and reflexive, symmetric q. relations

$$E_{B'} \subset B(L^2(X)) \;, \; \left( B = \mathcal{O}(X) \subset B(L^2(X)) \right)$$

**Conclusion**: There are plenty!

## $G_X$, for $X = (\mathcal{O}(X), \psi_X, A_X)$ - Q Graph

**Categorical**: $\mathcal{O}(G_X)$ - fund $\overset{\text{unitary}}{\text{rep}}$ $U = [u_{ij}]$, making the map

$$\rho_X : \mathcal{O}(X) \longrightarrow \mathcal{O}(X) \otimes \mathcal{O}(G_X)$$

$$e_i \longmapsto \sum_k e_k \otimes u_{kj}$$

a - unital $*$-homomorphism

- $\psi_X$-preserving : $(\psi_X \otimes \iota)\rho_X(\cdot) = \psi_X(\cdot)1$
- $A_X$-covariant : $\rho_X(A_X \cdot) = (A_X \otimes \iota)\rho_X(\cdot)$

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$$

$$G_X = (\mathcal{O}(G_X), \Delta) \quad - \quad \text{CMQG}$$

**Fact**:
- $U$ unitary
- $(A_X \otimes \iota)U = U(A_X \otimes 1)$
- $U(\eta \otimes 1) = \eta \otimes 1$
- $(m \otimes 1)(U \otimes U) = U(m \otimes 1)$

→ for $X$ classical recover usual $G_X$!

# Quantum Isomorphisms

$X, Y$ - graphs

$\mathrm{iIso}(X,Y)$ = space of isomorphisms $X \Longleftrightarrow Y$

~~$\cong \{ P = [p_{ij}] \in M_{|X|,|Y|}(\mathbb{C})$ permutation~~ $\Big|$ $X \ni x \longmapsto \varphi x \in Y$

$y \in Y \longmapsto \Phi(y)$

## Quantum Analogue

get map $\rho_{\cancel{X,Y}\; Y,X} : \mathcal{O}(X) \longrightarrow \mathcal{O}(Y) \otimes \mathcal{O}(\mathrm{iIso}(X,Y))$

$$\rho_{X,Y}(f)(y, \Phi) = f(\Phi(y))$$

and $\rho_{Y,X}(A_X f) = (A_Y \otimes \iota)\rho_{Y,X}(f)$

**Def'n:** Q. Isomorphism space $\mathrm{Iso}(X,Y)$:

universal unital $*$-alg. generated by $\mathcal{O}(\mathrm{Iso}(X,Y))$

coefficients of the unital $*$-homo

$$\rho_{Y,X} : \mathcal{O}(X) \to \mathcal{O}(Y) \otimes \mathcal{O}(\mathrm{Iso}(X,Y))$$

s.t. $(\psi_Y \otimes \iota)\rho_{Y,X} = \psi_X(\cdot)1$

$$\rho_{Y,X}(A_X \cdot) = (A_Y \otimes \iota)\rho_{Y,X}$$

**Theorem** (BCEHPSW): If $\underset{\underset{Z}{\shortparallel}}{\mathcal{O}(\mathrm{Iso}(X,Y))}$ is non zero,

then $Z$ admits the structure of a measured

$\mathcal{O}(G_Y) - \mathcal{O}(G_X)$ - Bigalois Extension

In particular, $G_X \sim G_Y$ via a mon eq.

$$F : \mathrm{Rep}(G_X) \to \mathrm{Rep}(G_Y)$$

$$u_X \longmapsto u_Y$$

$$A_X \longrightarrow A_Y$$

**Conversely**: Spo $X$ given, $G_X \cancel{\cong} G$.

and $F : \mathrm{Rep}(G_X) \to \mathrm{Hilb}_f$ is unitary f.f.

~~Then F [illegible] ∃ unitary fiber functor~~

Then $G_F \cong G_Y$

$\mathcal{O}(Y) = F(u)$

$m_Y = F(m_X)$, , $A_Y = F(A_X)$

$\eta_Y = F(\eta_X)$,

$\eta^*_Y = F(\eta^*_X)$,