Book recommendation:

The Gamma function

By Emil Artin

I have known about Euler’s gamma function for many years but none of the courses I took back in the 1980’s contained any discussion of this function. One could perhaps say that it was too complicated to include in the first courses and too specific to include in the later courses.

The present monograph (from 1964) is an English translation of Emil Artin’s book (written in German and dating back to 1931). It is written in a very elegant and certainly classic style – and consists of only 39 pages. In the preface he clearly describes the purpose of writing the book, and he also puts up some limitations, most notably that extension to complex variables will not be discussed. Hence not much more than the first year curriculum in our department will suffice as a background for reading the book. When reading it I occasionally had to take a small break and admire the elegance and rigor of the exposition!

No exposition about the Gamma function without the celebrated theorem of Bohr and Mollerup! This result – proved on pages 14-15 – states that the gamma function is the only positive function f defined on the positive real axis which satisfies the functional equation \( f(x+1)=xf(x) \), \( f(1)=1 \), and for which \( \log f(x) \) is convex.

You may notice the Danish touch – Harald Bohr (1887-1951) was a professor at Polyteknisk Lærenanstalt (now Technical University of Denmark) and later at University of Copenhagen, and Johannes Mollerup (1872-1937) was a professor at Polyteknisk Læreanstalt. Their monograph “Lære bog i matematisk Analyse” from 1922 contained the ideas behind the result. (Recently, I had the privilege to read in this rather old book too!) There are other connections to Danish mathematicians; In the preface to Artin’s book the editor writes that Børge Jessen, a student of Harald Bohr, observed a small error. (However I don’t know which formula the editor is actually referring to!)

I mentioned that Artin does not treat extension to complex variables. This is of course possible and the Gamma function fits well into the theory of holomorphic functions. Within this more general framework Wielandt proved another beautiful characterization of the Gamma function: it is the only holomorphic function f in the right half plane that satisfies the functional equation, \( f(1)=1 \), and which is bounded in the vertical strip \( x+iy \), where \( 1<x<2 \). Let me end this paragraph by mentioning that the assumption on boundedness can be weakened – and this is due to Bent Fuglede (about 10 years ago). Bent Fuglede is another prominent Danish mathematician.

The exposition is classic: no colors, no illustrations. Well, except for the colorful front page (probably from 2015) where one can actually see (in red) the graph of the Gamma function. But nowadays graphics are only a few Maple commands away.

I recommend this book to everyone wanting to know more about this world famous function. It is easy to read, extremely elegantly written – and not too long.

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