Talks on prismatic cohomology and the computation of $K_*(\mathbb{Z}/p^n)$

Masterclass "Topological Hochschild homology and zeta values"

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February 4, 2023

The following are notes of a series of five talks given as part of the masterclass "Topological Hochschild homology and zeta values" at Copenhagen university, mainly on recent results obtained in joint work with Ben Antieau and Thomas Nikolaus. Video recordings of the lectures should also be available on YouTube.

1 Prismatic cohomology

Prismatic cohomology is a cohomology theory of rings (and schemes) whose discovery is based on the behaviour of topological Hochschild homology. Let us quickly review this story:

For a ring R, the topological Hochschild homology is a spectrum given by

$$\mathrm{THH}(R) = R \otimes_{R \otimes_{\mathbb{S}} R^{\mathrm{op}}} R$$

Theorem 1.1 (Bökstedt)**.** *For a perfect $\mathbb{F}_p$-algebra R, we have*

$$\mathrm{THH}_*(R) = R[x],$$

with x in degree 2.

$\mathrm{THH}(R)$ carries an action by S^1, allowing us to form

$$\mathrm{TC}^-(R) := \mathrm{THH}(R)^{hS^1}, \quad \mathrm{TP}(R) := \mathrm{THH}(R)^{tS^1}$$

Looking at the homotopy fixed point and Tate spectral sequences, they take the following form:

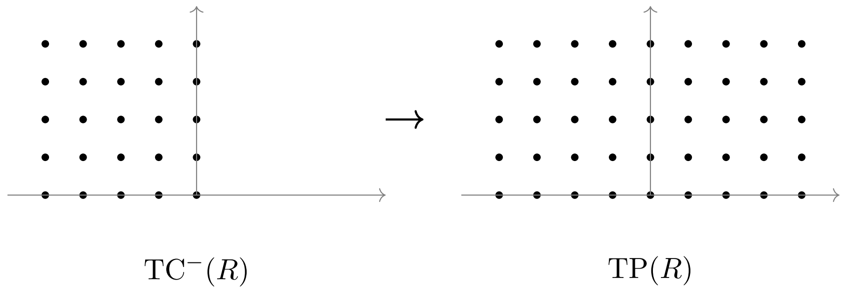

Figure 1: A qualitative picture of the homotopy fixed point and Tate spectral sequences for $\mathrm{TC}^-(R)$ and $\mathrm{TP}(R)$, and the can map between them.

Here every dot is a copy of R, and they live in even degrees. As a ring, everything is generated by x in degree $(0, 2)$ and t in degree $(-2, 0)$, where we also have t^{-1} in the case of TP.

Everything degenerates, and we have $\mathrm{TP}_0 \cong \mathrm{TP}_{2i}$ (noncanonically!). It turns out that TP_0 is the most nontrivial extension of R's one can build, the Witt vectors. For a perfect $\mathbb{F}_p$-algebra R, $W(R)$ is the unique p-complete, p-torsion free $\mathbb{Z}_p$-module lifting R. Note that we may recover TC^-_{2i} from the filtration on TP_{2i}: Writing $\mathrm{N}^{\geq i}$ for the (double speed) filtration, we have $\mathrm{TC}^-_{2i}(R) = \mathrm{N}^{\geq i}\,\mathrm{TP}_{2i}(R)$.

We have one more piece of information on $\mathrm{THH}(R)$, namely the *cyclotomic Frobenius*, an equivariant map $\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$. On homotopy fixed points, this induces

$$\mathrm{TC}^-(R) \to (\mathrm{THH}(R)^{tC_p})^{hS^1} \simeq \mathrm{THH}(R)^{tS^1} = \mathrm{TP}(R).$$

This is not compatible with the spectral sequences above, but instead with the homotopy fixed point spectral sequence for the S^1-action on $\mathrm{THH}(R)^{tC_p}$, i.e. we have:

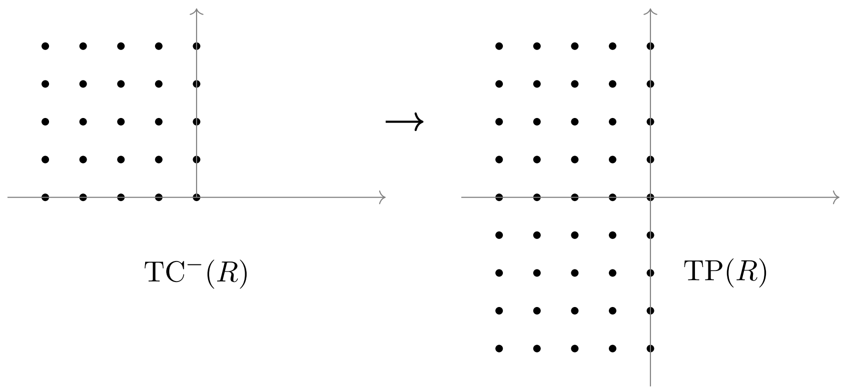

Figure 2: The Frobenius $\varphi : \mathrm{TC}^- \to \mathrm{TP}$, where the spectral sequence on the right is the homotopy fixed point spectral sequence for $\mathrm{TP}(R) = (\mathrm{THH}(R)^{tC_p})^{hS^1}$.

Since we must have $p \mapsto p$, we see that φ gives an injective map

$$\mathrm{N}^{\geq i} \mathrm{TP}_{2i}(R)/\mathrm{N}^{\geq i+1} \mathrm{TP}_{2i}(R) \to \mathrm{TP}_{2i}(R)/p.$$

Everything in sight is determined by the following data:

1. We have a ring $\mathrm{TP}_0(R)$ with a filtration $\mathrm{N}^{\geq *} \mathrm{TP}_0(R)$, and invertible modules $\mathrm{TP}_{2i}(R)$ over it. $\mathrm{TP}_0(R)$ is some kind of torsion free lift of R.
2. We have a Frobenius map $\mathrm{N}^{\geq i} \mathrm{TP}_{2i}(R) \to \mathrm{TP}_{2i}(R)$. In particular, we have some kind of Frobenius endomorphism on $\mathrm{TP}_0(R)$.

The above only works for special R: Perfect $\mathbb{F}_p$-algebras, as we have seen, as well as perfectoid rings, and more generally regular quotients of those. The idea behind prismatic cohomology is to set up an algebraic version of this situation that works more generally.

Definition 1.2. *A δ-ring (with respect to a fixed prime p) is a ring A together with a map $\delta : A \to A$ with:*

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}$$
$$\delta(xy) = x^p \delta(y) + y^p \delta(x) + p\delta(x)\delta(y).$$

These relations imply that $\varphi(x) := x^p + p\delta(x)$ is a ring endomorphism of A, lifting the Frobenius on A/p.

In fact, on a p-torsion free ring A, δ-ring structures are the same as Frobenius lifts, since $\delta(x) = \frac{\varphi(x) - x^p}{p}$.

Example 1.3.

1. $\mathbb{Z}$ or $\mathbb{Z}_p$ with φ given by the identity,
2. $W(R)$ with φ given by the Witt vector Frobenius,
3. $\mathbb{Z}_p[z]$ with $\varphi(z) = z^p$.

Definition 1.4. *A prism is a δ-ring A together with an ideal $I \subseteq A$ such that:*

1. *I is an invertible A-module (i.e. locally principal), and A is (p, I)-complete,*
2. *I is locally generated by a distinguished element, i.e. by a d such that $\delta(d)$ is a unit.*

If I is (globally) principal, we will call A orientable, and a choice of generator $d \in I$ an orientation. This d is then automatically distinguished. This is the situation that we will usually be in.

Example 1.5.

1. $(\mathbb{Z}_p, (p))$ (and no other ideal works here!),
2. more generally, $(W(R), (p))$,
3. $(\mathbb{Z}_p[[z]], (z + p))$, or any other Eisenstein polynomial.

We call a prism *crystalline* if $I = (p)$, and *perfect* if its Frobenius lift is invertible. Here is a nice table of different bijective correspondences:

A	A/I
perfect, crystalline	perfect $\mathbb{F}_p$-algebra
perfect	perfectoid ring
$W(k)[[z]]$	discrete valuation ring with residue field k and chosen uniformizer

We are now ready to define prismatic cohomology:

Definition 1.6. *Given a prism (A, I) and an A/I-algebra R, the* prismatic site $\mathbb{\Delta}(R/A)$ *of (A, R) is given by the category whose objects are prisms (B, J) with a diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ R & \longrightarrow & B/J \end{array}$$

where $A \to B$ is a map of prisms, i.e. a map of δ-rings taking I into J (it can be checked that then automatically $J = BI$).

We declare $(B, J) \to (B', J')$ a cover if $B \to B'$ is (p, I)-completely faithfully flat, and define sheaves $\mathcal{O}_{\mathbb{\Delta}(R/A)}$ and $\overline{\mathcal{O}}_{\mathbb{\Delta}(R/A)}$ taking (B, J) to B and B/J, respectively.

Definition 1.7. *We define*

$$\mathbb{\Delta}_{R/A} := R\Gamma(\mathcal{O}_{\mathbb{\Delta}(R/A)}) = \lim_{(B,J)\in\mathbb{\Delta}(R/A)} B$$
$$\overline{\mathbb{\Delta}}_{R/A} := R\Gamma(\overline{\mathcal{O}}_{\mathbb{\Delta}(R/A)}) = \lim_{(B,J)\in\mathbb{\Delta}(R/A)} B/J.$$

(Derived global sections can be viewed as maps of the constant presheaf into the derived sheafification. By faithfully flat descent, these are already derived sheaves, so we simply need maps from the constant presheaf, which can be written as limit over the category as above.)

Example 1.8. If $R = A/I$, then the prismatic site has an initial object, namely (A, I) itself. So

$$\mathbb{\Delta}_{(A/I)/A} = A, \quad \overline{\mathbb{\Delta}}_{(A/I)/A} = A/I$$

2 Prismatic envelopes

Generally, $\mathbb{\Delta}_{R/A}$ and $\overline{\mathbb{\Delta}}_{R/A}$ can have negative homotopy. To describe this, observe that every prism B is filtered by powers J^i, with associated graded J^i/J^{i+1}. This gives a filtration on $\mathcal{O}_{\mathbb{\Delta}(R/A)}$ whose associated graded terms are given by $\overline{\mathcal{O}}_{\mathbb{\Delta}(R/A)}\{i\} := \overline{\mathcal{O}}_{\mathbb{\Delta}(R/A)} \otimes_A I^i$.

After global sections, we get a filtration on $\Delta_{R/A}$, the *Hodge-Tate filtration*, with associated graded terms

$$\overline{\Delta}_{R/A}\{i\} := \overline{\Delta}_{R/A} \otimes_A I^i,$$

and we get connecting homomorphisms ("Bocksteins")

$$d : \overline{\Delta}_{R/A}\{i\} \to \overline{\Delta}_{R/A}\{i+1\}[1].$$

Note that by definition we have a map $R \to \overline{\Delta}_{R/A}$, making $\bigoplus_i \pi_{-i}\overline{\Delta}_{R/A}\{i\}$ a cdga over R. This is strict, and we have

Theorem 2.1 (Hodge-Tate comparison, Thm. 4.11 in [**?**])**.** *If R is smooth over A/I, then the canonical map*

$$\Omega^*_{R/(A/I)} \to \pi_{-*}\overline{\Delta}_{R/A}\{*\}$$

is an isomorphism. In particular,

$$\pi_{-i}\overline{\Delta}_{R/A} \cong \Omega^i_{R/(A/I)} \otimes_A I^{-i}.$$

This means that in the smooth case, all twists of $\overline{\Delta}_{R/A}$, and hence also $\Delta_{R/A}$, are concentrated in a range of dimensions $[-n, 0]$, where n is the dimension of $\Omega^*_{R/(A/I)}$.

We now want to replace $\Delta_{R/A}$ by its left Kan extension from smooth algebras (as a functor to (p, I)-complete A-modules). This does not change its value for R quasisyntomic over A/I, but ensures that the following holds generally.

Theorem 2.2 (Derived Hodge-Tate comparison)**.** *There is an ascending filtration $\mathrm{F}^{\mathrm{conj}}_{\leq i}\overline{\Delta}_{R/A}$ with i-th associated graded*

$$\mathrm{gr}^{\mathrm{conj}}_i \overline{\Delta}_{R/A} = \mathrm{L}\Omega^i_{R/(A/I)}[-i] \otimes_A I^{-i}.$$

If R is a quotient of A/I by a regular sequence, or more generally $L_{R/(A/I)}$ has p-complete Tor amplitude in $[1, 1]$, then one can check that all $L\Omega^i_{R/(A/I)}$ are concentrated in degree i. So the associated graded above will be in degree zero! Thus, in this case, $\Delta_{R/A}$ and $\overline{\Delta}_{R/A}$ are discrete.

We have not yet discussed the Frobenius. Since each B comes with a Frobenius lift, $\mathcal{O}_{\Delta(R/A)}$ has a φ_A-semilinear endomorphism φ. Thus, we get the *absolute Frobenius*

$$\varphi : \Delta_{R/A} \to \Delta_{R/A},$$

which is φ_A-semilinear. We may view this also as an A-linear morphism

$$\varphi_{R/A} : \Delta^{(1)}_{R/A} := \Delta_{R/A} \otimes_{A,\varphi} A \to \Delta_{R/A}.$$

In the case where $\Delta_{R/A}$ is discrete, it makes sense to pull back the Hodge-Tate (i.e. I-adic) filtration from $\Delta_{R/A}$ to $\Delta^{(1)}_{R/A}$ along this map. We obtain a filtration on $\Delta^{(1)}_{R/A}$ defined by:

$$\mathrm{N}^{\geq i}\Delta^{(1)}_{R/A} = \{x \in \Delta^{(1)}_{R/A} \mid \varphi_{R/A}(x) \in I^i\}.$$

This is the Nygaard filtration. For example, elements of $\Delta^{(1)}_{R/A} = \Delta_{R/A} \otimes_{A,\varphi} A$ of the form $1 \otimes d$ with $d \in I$ lie in $\mathrm{N}^{\geq 1}$.

However, the maps on associated gradeds

$$\mathrm{gr}^i_{\mathrm{N}} \Delta^{(1)}_{R/A} \to \overline{\Delta}_{R/A}\{i\}$$

are not surjective, so the pullbacks defining the Nygaard filtration are not derived. It is however possible to extend from the discrete to the general case using $\Delta_{R/A}$ quasisyntomic descent in R. We will take the existence of such a Nygaard filtration on $\Delta^{(1)}_{R/A}$ for granted. The image above in fact can be identified with $\mathrm{F}^{\mathrm{conj}}_{\leq i}\overline{\Delta}_{R/A}\{i\}$, so that the relative Frobenius generally induces an equivalence

$$\mathrm{gr}^i_{\mathrm{N}} \Delta^{(1)}_{R/A} \to \mathrm{F}^{\mathrm{conj}}_{\leq i}\overline{\Delta}_{R/A}\{i\}$$

Note that in the smooth case, this means that $\mathrm{gr}^i_{\mathrm{N}} \Delta^{(1)}_{R/A} \cong \tau_{\geq -i}\overline{\Delta}_{R/A}\{i\}$. This suggests the following picture:

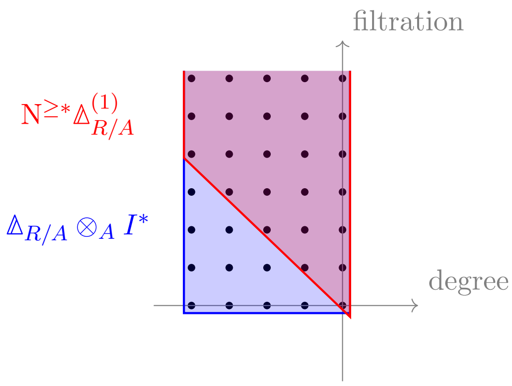

Figure 3: The relative Frobenius between Nygaard-filtered Frobenius-twisted prismatic cohomology and Hodge-Tate filtered prismatic cohomology, for R smooth over A/I (in the picture of dimension 4)

In the discrete case that we're about to discuss, all of the horizontal direction is mushed together into one homotopical degree, but it still corresponds to the conjugate filtration $\mathrm{F}^{\mathrm{conj}}_{\leq *}\overline{\Delta}_{R/A}$.

Now for simplicity, assume we have an oriented prism (A, d), and $R = A/(d, r)$ is a quotient of A/d by a nonzerodivisor. An object of the prismatic site $\Delta(R/A)$ is a prism (B, J) with a diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ R & \longrightarrow & B/J \end{array}$$

i.e. a prism with a map $A \to B$ such that r is mapped into J.

Lemma 2.3. *If A is a δ-ring, and we have $d = ud'$ with*

1. *$\delta(d)$ is a unit,*
2. *A is complete with respect to (p, d'),*

then u and $\delta(d')$ are also units.

Proof. Apply δ to get

$$\delta(d) = u^p\delta(d') + \delta(u)(d')^p + p\delta(u)\delta(d'),$$

i.e. $\delta(d) = u^p\delta(d') \bmod (p, d')$. It follows that $u^p\delta(d')$, and hence u, is a unit. $\square$

In particular we see that in the situation above the lemma, d automatically generates $J \subseteq B$, since locally J is assumed to be generated by a nonzerodivisor (this is the claim we made in the previous lecture, that a morphism of prisms automatically satisfies $J = B \cdot I$).

So we see that objects of $\Delta(R/A)$ are exactly the (p, d)-complete δ-rings under A in which r is divisible by d.

The following lemma is not hard to check:

Lemma 2.4.
1. *If A is a δ-ring, there exists a free δ-ring under A on one generator, and it is of the form*
$$A\{x\} := A[x, \delta(x), \ldots].$$
2. *If A is a δ-ring and $r \in A$ an element, there exists an initial δ-ring under A with $r = 0$, and it is of the form*
$$A/(r)_\delta := A/(r, \delta(r), \ldots).$$

Corollary 2.5 (Prismatic Envelopes)**.** *For $R = A/(d, r)$ as above, there is an initial object in $\Delta(R/A)$ of the form*

$$\Delta_{R/A} = A\left\{\frac{r}{d}\right\}^{\wedge}_{(p,d)} = (A\{a\}/(da - r)_\delta)^{\wedge}_{p,d}$$

We may also give a similar formula for the Frobenius twist, since

$$\Delta_{R/A} \otimes_{A,\varphi} A = (A \otimes_{A,\varphi} A)\left\{\frac{r \otimes 1}{d \otimes 1}\right\}^{\wedge}_{(p,d\otimes 1)} = A\left\{\frac{\varphi(r)}{\varphi(d)}\right\}^{\wedge}_{(p,d)}$$

where in the last step we've used that $(p, \varphi(d)) = (p, d^p)$ to rewrite the $(p, \varphi(d))$-completion as (p, d)-completion. In what follows, we will always take this perspective on $\Delta^{(1)}_{R/A}$, in particular we will view it as A-module via the new (base-changed) A-module structure. So when we write $r \in \Delta^{(1)}_{R/A}$, we mean $1 \otimes r$ (and $r \otimes 1$ we simply write as $\varphi(r)$).

3 Explicit generators for prismatic envelopes

In the previous talk, we saw that for a regular quotient $R = A/(d, r)$, prismatic cohomology has a description as prismatic envelope,

$$\Delta_{R/A} = A\left\{\frac{r}{d}\right\}^{\wedge}_{p,d}$$

In particular, $\Delta_{R/A}$ is itself a prism here (with divisor (d)). It differs from A in two important respects:

1. We have produced new elements $a, \delta a, \ldots$ (subject to some relations), where $a = \frac{r}{d}$.
2. We have also caused r to lie in $\mathrm{N}^{\geq 1}\Delta^{(1)}_{R/A}$. Indeed, the relative ($A$-linear) Frobenius takes $\varphi_{R/A}(r) = r = ad \in I^1$. Recall that we said earlier that the Frobenius $\mathrm{gr}^1_{\mathrm{N}}\,\Delta^{(1)}_{R/A} \to \overline{\Delta}_{R/A}\otimes_A I^i$ has image $\mathrm{F}^{\mathrm{conj}}_{\leq i}\overline{\Delta}_{R/A}\otimes_A I^i$, so this shows that $a \in \mathrm{F}^{\mathrm{conj}}_{\leq 1}\overline{\Delta}_{R/A}$.

The relation $da - r$, upon application of δ, turns into:

$$\delta(d)a^p + \varphi(d)\delta(a) + \text{other terms} = 0,$$

which, since $\delta(d)$ is a unit, yields a relation of the form

$$a^p = -p\delta(a) + \ldots$$

Recall that the associated graded of $\mathrm{F}^{\mathrm{conj}}_{\leq *}\overline{\Delta}_{R/A}$ is given by $L\Omega^*_{R/(A/I)}[-*]$, which in the case of $R = A/(d, r)$ is a free divided power algebra. One would hope that the $\delta^u a$ play the role of p-typical divided powers, at least up to the conjugate filtration. In fact, we have the following:

Proposition 3.1. *For $R = A/(d, r)$, the R-module $\overline{\Delta}_{R/A}$ is free on monomials $\prod_u (\delta^u a)^{e_u}$, where all $e_u < p$.*

What we will actually need to understand though is the Nygaard filtration on $\Delta^{(1)}_{R/A}$. In principle, the above choice of generators $\delta^u(a)$ gives generators for the Frobenius twist as well, but they don't play well with the Nygaard filtration.

In order to understand the Nygaard filtration on the Frobenius twist, recall that we would expect $\delta^u a \in F^{\mathrm{conj}}_{\leq p^u}$ (from the divided power claim), so we expect there to be elements in $\mathrm{N}^{\geq p^u}\Delta^{(1)}_{R/A}$ mapping to those $\delta^u a$, i.e. elements whose image under the relative Frobenius is divisible by d^{p^u}.

We write the absolute Frobenius on $\Delta^{(1)}_{R/A}$, i.e. the composite

$$\Delta^{(1)}_{R/A} \xrightarrow{\varphi_{R/A}} \Delta_{R/A} \to \Delta^{(1)}_{R/A}$$

simply as φ. Note that if an element becomes divisible by d^i under the relative Frobenius, it becomes divisible by $\varphi(d)^i$ under the absolute Frobenius, since the inclusion on the right is φ_A-semilinear.

Definition 3.2. *We define an operation*

$$\delta_i : \mathrm{N}^{\geq i}\Delta^{(1)}_{R/A} \to \mathrm{N}^{\geq pi}\Delta^{(1)}_{R/A}$$

by

$$\delta_i(x) = \delta(x) - \delta(d^i)\frac{\varphi(x)}{\varphi(d^i)}.$$

Lemma 3.3. *The δ_i are well-defined, define a δ-ring structure on the graded ring $\bigoplus \mathrm{N}^{\geq i}\Delta^{(1)}_{R/A}$, and satisfy*

$$\begin{array}{ccc} \mathrm{N}^{\geq i}\Delta_{R/A} & \xrightarrow{\varphi_{R/A}/d^i} & \Delta_{R/A} \\ \downarrow{\scriptstyle \delta_i} & & \downarrow{\scriptstyle \delta} \\ \mathrm{N}^{\geq pi}\Delta_{R/A} & \xrightarrow{\varphi_{R/A}/d^{pi}} & \Delta_{R/A} \end{array}$$

Proof. If everything is p-torsion free, observe that $\widetilde{\varphi}$, defined on the summand $\mathrm{N}^{\geq i}\Delta^{(1)}_{R/A}$ by

$$\widetilde{\varphi}(x) = d^{pi}\frac{\varphi(x)}{\varphi(d)^i}$$

is a Frobenius lift with associated δ-ring structure given by the δ_i. Indeed, it agrees with:

$$\begin{aligned} &(\varphi(d^i) - p\delta(d^i))\frac{\varphi(x)}{\varphi(d)^i} \\ =&\varphi(x) - p\delta(d^i)\frac{\varphi(x)}{\varphi(d)^i} \\ =&x^p + p(\delta(x) - \delta(d^i)\frac{\varphi(x)}{\varphi(d)^i}). \end{aligned}$$

Also, the diagram

$$\begin{array}{ccc} \mathrm{N}^{\geq i}\Delta_{R/A} & \xrightarrow{\varphi_{R/A}/d^i} & \Delta_{R/A} \\ \downarrow{\scriptstyle \widetilde{\varphi}} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{N}^{\geq pi}\Delta_{R/A} & \xrightarrow{\varphi_{R/A}/d^{pi}} & \Delta_{R/A} \end{array}$$

commutes, since

$$\frac{1}{d^{pi}}\varphi_{R/A}\left(d^{pi}\frac{\varphi(x)}{\varphi(d)^i}\right) = \frac{\varphi_{R/A}\varphi(x)}{\varphi(d)^i} = \varphi\left(\frac{\varphi_{R/A}(x)}{d^i}\right).$$

The non-torsion free case follows from this by a basechange argument (the universal example has R torsion free). ∎

This gives us a way to describe generators of the Frobenius twist that play well with the Nygaard filtration. Indeed, we have a starting element $r \in \mathrm{N}^{\geq 1}\Delta^{(1)}_{R/A}$, and can recursively define a sequence $f_i \in \mathrm{N}^{\geq p^i}\Delta^{(1)}_{R/A}$ with $f_0 = r$, $f_{i+1} = \delta_{p^i} f_i$.

As an immediate observation, note that $\varphi_{R/A}(f_i) = (\delta^i a)d^{p^i}$, and so since $\varphi_{R/A} : \mathrm{N}^{\geq i}\Delta^{(1)}_{R/A} \to \overline{\Delta}_{R/A} \otimes_A I^i$ has image $\mathrm{F}^{\mathrm{conj}}_{\leq i}\overline{\Delta}_{R/A} \otimes_A I^i$, we learn that $\delta^i a \in \mathrm{F}^{\mathrm{conj}}_{\leq p^i}\overline{\Delta}_{R/A}$ as claimed.

Theorem 3.4. *If $R = A/(d, r)$ as above, then*

1. *an R-basis for $\overline{\Delta}_{R/A}$ is given by monomials $\prod_u (\delta^u a)^{e_u}$, with conjugate filtration determined by $\delta^u a \in \mathrm{F}^{\mathrm{conj}}_{\leq p^u}$,*
2. *an R-basis for $\mathrm{gr}^*_{\mathrm{N}} \Delta^{(1)}_{R/A}$ is given by monomials $d^c \prod_u f_u^{e_u}$, with $e_u < p$, with d in degree 1 and f_u in degree p^u,*
3. *the isomorphism*
$$\mathrm{gr}^n_{\mathrm{N}} \Delta^{(1)}_{R/A} \to \mathrm{F}^{\mathrm{conj}}_{\leq n}\overline{\Delta}_{R/A}$$
induced by $\varphi_{R/A}/d^n$, takes $d^c \prod_u f_u^{e_u} \mapsto \prod_u (\delta^u a)^{e_u}$.

For $R = A/(d, r_1, \ldots, r_n)$ we have a similar formula with multiple a_v and f_{uv}.

Note that we have preferred lifts of the $\delta^u a$ and f_u to $\Delta_{R/A}$ and $\Delta^{(1)}_{R/A}$, and in principle we know how to compute the relations. We now want to get a more explicit picture of this in the case of $A = W(k)[[z]]$, with $d = E(z)$ the Eisenstein polynomial of some purely ramified extension $\mathcal{O}_K$ of $W(k)$. Hence we have $A/(d) = \mathcal{O}_K$, and can write $R = \mathcal{O}_K/\varpi^n$ as a regular quotient $A/(d, r)$ with $r = z^n$. In this case, the pair (A, R) admits another filtration:

Definition 3.5. *A filtered δ-ring is a δ-ring A with a multiplicative filtration $\mathrm{F}^{\geq \star}A$ with*

$$\delta(\mathrm{F}^{\geq i}A) \subseteq \mathrm{F}^{\geq pi}A$$

Note that this implies $\varphi(\mathrm{F}^{\geq i}A) \subseteq \mathrm{F}^{\geq pi}A$ (and is equivalent to it if A and $\mathrm{gr}^*_{\mathrm{F}} A$ are torsion free).

Example 3.6. The δ-ring $W(k)[[z]]$ with the z-adic filtration defines a filtered δ-ring, since $\delta(z) = 0 \in \mathrm{F}^{\geq 1}$.

The compatible filtration on $\mathcal{O}_K/\varpi^n = W(k)[[z]]/(d, z^n)$ is the ϖ-adic filtration. We get an induced filtration on the prismatic envelope (and in fact, one can see that prismatic cohomology for such a filtered pair inherits a filtration in general), where in $\Delta^{(1)}_{R/A}$ we have:

$$f_0 = r = z^n \in \mathrm{N}^{\geq 1}\Delta^{(1)}_{R/A} \cap \mathrm{F}^{\geq n}\Delta^{(1)}_{R/A}$$
$$f_1 = \delta_1 f_0 \in \mathrm{N}^{\geq p} \cap \mathrm{F}^{\geq pn}$$
$$\dots$$
$$f_u \in \mathrm{N}^{\geq p^u} \cap \mathrm{F}^{\geq p^u n}$$

We thus have the following picture:

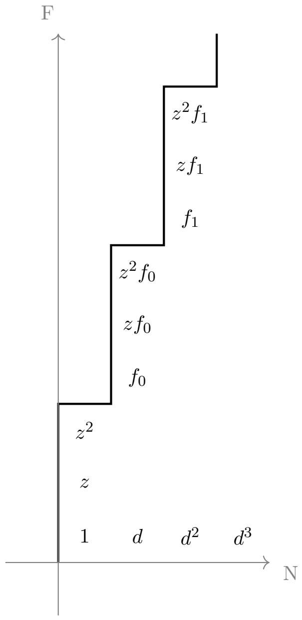

Figure 4: Picture of various elements of $\Delta^{(1)}_{R/A}$ for $A = W(k)[[z]]$ and $R = \mathcal{O}_K/\varpi^n$, with $n = 3$ and $p = 2$, with respect to the N and F filtrations.

This staircase pattern immediately allows us to deduce that the F-filtration is bounded on $\Delta^{(1)}_{R/A}/\mathrm{N}^{\geq i}\Delta^{(1)}_{R/A}$. In the next lecture we will see how to leverage this to reduce the computation of $K(\mathcal{O}_K/\varpi^n)$ to a finite computation.

4 An algorithm to compute $K(\mathcal{O}_K/\varpi^n)$

Let $A = W(k)[[z]]$, $R = \mathcal{O}_K/\varpi^n = W(k)[[z]]/(E(z), z^n)$, $d = E(z)$.

Let us draw some conclusions from the picture with the F and N filtrations:

We know that $\mathrm{gr}^*_{\mathrm{N}}\,\Delta^{(1)}_{R/A}$ is free as R-module on the monomials $d^c \prod (f_u)^{e_u}$ with $e_u < p$. So $\mathrm{gr}^{*,*}_{\mathrm{F,N}}\,\Delta^{(1)}_{R/A}$ is free as k-module on monomials $z^k d^c \prod (f_u)^{e_u}$, with $k < n$. Since in

$\mathrm{gr}^*_{\mathrm{F}}\, \Delta^{(1)}_{R/A}$, $d = E(z) = E(0)$ is a unit multiple of p, this means that $\mathrm{gr}^*_{\mathrm{F}}$ is free as a $W(k)$-module on monomials $z^k \prod (f_u)^{e_u}$ with $k < n$, $e_u < p$. So each row in our picture is a single copy of $W(k)$.

Lemma 4.1. *A $W(k)$-basis for $\mathrm{N}^{\geq i} \Delta^{(1)}_{R/A}$ is given by those monomials $z^k d^c \prod (f_u)^{e_u}$ with $k < n$, $e_u < p$, and c minimal such that $c + \sum p^u e_u \geq i$.*

So the $\mathrm{gr}^*_{\mathrm{F}}$ of everything in sight is free of rank 1 on $W(k)$. Let us now call a map a *filtered isogeny* if it is a rational isomorphism on $\mathrm{gr}^* \mathrm{F}$ (and hence on each $\mathrm{F}^{[a,b]}$).

Corollary 4.2. *Let $A = W(k)[[z]]$, $R_n = \mathcal{O}_K/\varpi^n$, and $R_\infty = \mathcal{O}_K$. The following are filtered isogenies:*

1. *The map* $\mathrm{can} : \mathrm{N}^{\geq i} \Delta^{(1)}_{R_n/A} \to \Delta^{(1)}_{R_n/A}$.
2. *The map* $\Delta^{(1)}_{R_m/A} \to \Delta_{R_n/A}$ *for* $m \geq n$.

Proof. The only observation required is that z^i is nonzero in $\mathrm{gr}^i_{\mathrm{F}}\, \overline{\Delta}^{(1)}_{R_n/A}$. By computing the relations satisfied by the f_i^p, this follows. (We have not explained how to do this here, but one way is to start with the first relation and then apply the operations δ_i used to define the f_i). □

Now let us take a step back and think about what we want to compute. Recall from Ben's talk that $\Delta_{R/A}$ is actually functorial in pairs (A, R) of a δ-ring A and an A-algebra R, with no reference to a choice of prism structure on A, and that there is a descent diagram, heavily inspired by work of Liu-Wang on TC, of the form

$$\Delta_{R/\mathbb{Z}_p} \longrightarrow \Delta_{R/\mathbb{Z}_p[[z]]} \rightrightarrows \Delta_{R/\mathbb{Z}_p[[z_0,z_1]]} \Rrightarrow \cdots$$

The first term is *absolute prismatic cohomology*, the other terms can be identified with the prismatic envelopes we have been studying earlier, since $\mathbb{Z}_p \to W(k)$ is formally étale and hence $\Delta_{R/\mathbb{Z}_p[[z]]} = \Delta_{R/W(k)[[z]]}$. What we are interested in is

$$\mathbb{Z}_p(i)(R) = \mathrm{fib}\left(\mathrm{N}^{\geq i} \Delta^{(1)}_{R/\mathbb{Z}_p}\{i\} \xrightarrow{\mathrm{can} - \varphi} \Delta^{(1)}_{R/\mathbb{Z}_p}\{i\} \right)$$

Here $\{i\}$ is the Breuil-Kisin twist, which we will explain momentarily. For now, we only need to know that the descent diagram above works with any of those decorations, and so $\mathbb{Z}_p(i)(R)$ can be written as fiber of the map between totalisations:

$$\begin{array}{ccccc} \mathrm{N}^{\geq i} \Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} & \rightrightarrows & \mathrm{N}^{\geq i} \Delta^{(1)}_{R/\mathbb{Z}_p[[z_0,z_1]]}\{i\} & \Rrightarrow & \cdots \\ \downarrow{\scriptstyle \mathrm{can} - \varphi} & & \downarrow{\scriptstyle \mathrm{can} - \varphi} & & \\ \Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} & \rightrightarrows & \Delta^{(1)}_{R/\mathbb{Z}_p[[z_0,z_1]]}\{i\} & \Rrightarrow & \cdots \end{array}$$

As Ben explained in his talk, a priori results on cohomological dimension of absolute prismatic cohomology tell us that the rows have cohomology concentrated in H^0 and H^1. So we may replace the right hand terms by the kernel of the outgoing differential, and obtain a square

$$\begin{array}{ccc} \mathrm{N}^{\geq i}\Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} & \longrightarrow & \mathrm{N}^{\geq i}Z^{(1)}\{i\} \\ \big\downarrow{\scriptstyle \mathrm{can}-\varphi} & & \big\downarrow{\scriptstyle \mathrm{can}-\varphi} \\ \Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} & \longrightarrow & Z^{(1)}\{i\} \end{array}$$

quasi-isomorphic to $\mathbb{Z}_p(i)$. Now note also that each of the terms in the original complex has the staircase pattern $\mathrm{F}^{\geq nk} \subseteq \mathrm{N}^{\geq k}$. So after applying $\mathrm{F}^{\geq ni}$, the map

$$\mathrm{can} : \mathrm{F}^{\geq ni}\mathrm{N}^{\geq i}\Delta^{(1)}_{R/\mathbb{Z}_p[[z_\bullet]]}\{i\} \to \mathrm{F}^{\geq ni}\Delta^{(1)}_{R/\mathbb{Z}_p[[z_\bullet]]}\{i\}$$

becomes an isomorphism. Also, since φ strictly increases filtration, $\mathrm{can}-\varphi$ is still an isomorphism.

So the homotopy type of the fiber does not change if we truncate everything to

$$\begin{array}{ccc} \mathrm{F}^{[0,ni-1]}\mathrm{N}^{\geq i}\Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} & \longrightarrow & \mathrm{F}^{[0,ni-1]}\mathrm{N}^{\geq i}Z^{(1)}\{i\} \\ \big\downarrow{\scriptstyle \mathrm{can}-\varphi} & & \big\downarrow{\scriptstyle \mathrm{can}-\varphi} \\ \mathrm{F}^{[0,ni-1]}\Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} & \longrightarrow & \mathrm{F}^{[0,ni-1]}Z^{(1)}\{i\} \end{array}$$

Since $\mathrm{gr}^0_{\mathrm{F}}\,\mathbb{Z}_p(i)(R) = \mathbb{Z}_p(i)(k) = 0$ for $i > 0$, we may even replace everything by $\mathrm{F}^{[1,ni-1]}$.

What about the Breuil-Kisin twist $\{i\}$? For any prism A, there are invertible modules $A\{1\}$. These satisfy $A\{1\} \otimes_A A/I = I/I^2$, but they are not just given by I (like the twists we saw on Hodge-Tate cohomology), instead they are characterized by having $A\{1\} \otimes_A A/I_r = I_r/I_r^2$, where $I_r = I \cdot \varphi(I) \cdots \varphi^{r-1}(I)$, and can be written as limit of those. Their key property is that there is a natural Frobenius

$$\varphi : A\{i\} \to A\{i\} \otimes_A I^{-i}.$$

Taking the limit of the sheaf $\mathcal{O}_{\Delta(R/A)}\{i\}$ defines a corresponding twist on prismatic cohomology (relative δ-rings even, although then it isn't necessarily an invertible module). In the prismatic situation, we will make use of the observation that $A\{1\}$ is a free module of rank 1 over A if and only if I is, i.e. A is "Breuil-Kisin orientable" iff it is orientable. Given a "Breuil-Kisin orientation" s, there is a unique orientation d_s such that $\varphi(s) = s \otimes d_s^{-1}$. Hence if we just write $\Delta_{R/A}\{i\} = \Delta_{R/A} \cdot s^i$ in the prismatic case, we simply have to remember $\varphi(s) = \frac{s}{d_s}$. Similarly, $\Delta^{(1)}_{R/A}\{i\} = \Delta_{R/A}\{i\} \otimes_{A,\varphi} A$ is free on $s^i \otimes 1$, which under Frobenius gets divided by $\varphi(d_s)^{-i}$. (It turns out in order to be able to choose s so that d_s agrees with our previous choice of orientation, we have to normalize our Eisenstein polynomial with constant term p.)

So the left-hand terms in the square are free on monomials $z^k \prod f_u^{e_u}(s^i \otimes 1)$ and $z^k d^c \prod f_u^{e_u}(s^i \otimes 1)$, respectively.

What about the right hand term? $\Delta^{(1)}_{R/\mathbb{Z}_p[[z_0,z_1]]}$ has a prismatic envelope description, since $R = \mathbb{Z}_p[[z_0, z_1]]/(E(z_0), z_0^n, z_0 - z_1)$. The relation z_0^n gives rise to the f_i (or rather, their image under one of the two coboundary maps). The relation $z_0 - z_1$ gives rise to another sequence of generators, let us call those g_i.

By some explicit analysis of how these complexes look like on the level of F-associated gradeds (they look like a cobar complex of a divided power algebra), one can prove the following identification of the right-hand terms $Z^{(1)}$:

Lemma 4.3. *1.* $\Delta^{(1)}_{R/\mathbb{Z}_p[[z_0,z_1]]}$ *is a free* $\Delta^{(1)}_{R/\mathbb{Z}}$*-module on monomials* $\prod g_u^{e_u}$*.*

2. The module map $\Delta^{(1)}_{R/\mathbb{Z}_p[[z_0,z_1]]} \to \Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}$ *dual to the monomial* g_0 *(i.e. taking* $g_0 \mapsto 1$ *and all other* $\prod g_u^{e_u} \mapsto 0$*) restricts to an isomorphism* $\mathbb{Z}^{(1)} \to \Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}$ *of abelian groups, which shifts the filtrations* F *and* N *by* -1*.*

We think of the target free module as free on a generator ∇z, and interpret the shift of filtrations as having ∇z sit in $\mathrm{F}^{\geq 1} \cap \mathrm{N}^{\geq 1}$.

We thus arrive at the outer square in the diagram:

$$\begin{array}{ccccc}
\mathrm{F}^{[1,ni-1]}\mathrm{N}^{\geq i}\Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} & \longrightarrow & \mathrm{F}^{[1,ni-1]}\mathrm{N}^{\geq i}Z^{(1)}\{i\} & \longrightarrow & \mathrm{F}^{[0,ni-2]}\mathrm{N}^{\geq i-1}\Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} \cdot \nabla z \\
\downarrow{\scriptstyle \mathrm{can} - \varphi} & & \downarrow{\scriptstyle \mathrm{can} - \varphi} & & \downarrow{\scriptstyle \mathrm{can} - \varphi^{\nabla}} \\
\mathrm{F}^{[1,ni-1]}\Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} & \longrightarrow & \mathrm{F}^{[1,ni-1]}Z^{(1)}\{i\} & \longrightarrow & \mathrm{F}^{[0,ni-2]}\mathrm{N}^{\geq i-1}\Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} \cdot \nabla z
\end{array}$$

Here this identification is compatible with the can map in the obvious way, but φ^{∇} is not semilinear in any reasonable way, so one has to be a bit careful with thinking of this term as a module. (Maybe similar to the Frobenius on the de Rham Witt complex?)

We call the horizontal composite ∇ and observe that really $z \mapsto z_0 - z_1 = g_0 \mapsto \nabla z$ in the $i = 0$ case (in higher weight the BK orientation contributes, since $s_0 \otimes 1$ and $s_1 \otimes 1$ differ by some unit).

1. All terms are free $W(k)$-modules of rank $ni - 1$, i.e. free $\mathbb{Z}_p$-modules of rank $(ni - 1)f$, where f is the residue field degree of K, and we have described explicit bases, and can derive relations for reducing arbitrary monomials to that basis. Using this, we can get matrices for the can maps as well as for the maps induced by $\mathcal{O}_K \to \mathcal{O}_K/\varpi^n$.

2. Compute ∇ for $\mathcal{O}_K$. This is a map

$$\nabla : \Delta^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\} \to \Delta^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\} \cdot \nabla z$$

determined by taking $z^k(s\otimes 1)^i \mapsto z_0^k(s_0\otimes 1)^i - z_1^k(s_1\otimes 1)^i$, which we have to rewrite in terms of $z_0^c \prod g_u^{e_u}\cdot(s_0\otimes 1)^i$ and then extract the g_0 coefficient. We only need this through $\mathrm{F}^{[1,ni-1]}$, which is a finite problem (but one could compute more of it and share the result between different $\mathcal{O}_K/\varpi^n$).

3. Use the diagram

$$\begin{array}{ccc} \mathrm{N}^{\geq i}\mathbb{\Delta}^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\} & \xrightarrow{\nabla} & \mathrm{N}^{\geq i-1}\mathbb{\Delta}^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\}\cdot\nabla z \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\ \mathbb{\Delta}^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\} & \xrightarrow{\nabla} & \mathbb{\Delta}^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\}\cdot\nabla z \end{array}$$

and the fact that the vertical maps are filtered isogenies to compute the top horizontal map, through the range $\mathrm{F}^{[1,in-1]}$.

4. Now, use the diagram

$$\begin{array}{ccc} \mathrm{N}^{\geq i}\mathbb{\Delta}^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\} & \xrightarrow{\nabla} & \mathrm{N}^{\geq i-1}\mathbb{\Delta}^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\}\cdot\nabla z \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi^{\nabla}} \\ \mathbb{\Delta}^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\} & \xrightarrow{\nabla} & \mathbb{\Delta}^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\}\cdot\nabla z \end{array}$$

t similarly determine the right vertical map. The left vertical map is easy, it is determined by $d^i z^k(s\otimes 1)^i \mapsto z^{pk}(s\otimes 1)^i$.

5. Now we know the diagram

$$\begin{array}{ccc} \mathrm{N}^{\geq i}\mathbb{\Delta}^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\} & \xrightarrow{\nabla} & \mathrm{N}^{\geq i-1}\mathbb{\Delta}^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\}\cdot\nabla z \\ \downarrow{\scriptstyle \mathrm{can}-\varphi} & & \downarrow{\scriptstyle \mathrm{can}-\varphi^{\nabla}} \\ \mathbb{\Delta}^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\} & \xrightarrow{\nabla} & \mathbb{\Delta}^{(1)}_{\mathcal{O}_K/\mathbb{Z}_p[[z]]}\{i\}\cdot\nabla z \end{array}$$

through $\mathrm{F}^{[1-in]}$. Since all the maps induced by $\mathcal{O}_K \to \mathcal{O}_K/\varpi^n$ are filtered isogenies, we may infer the corresponding diagram for $\mathcal{O}_K/\varpi^n$, of which we now compute the cohomology of the total complex to obtain $H^*(\mathbb{Z}_p(i)(\mathcal{O}_K/\varpi^n)$.

In the above description, we have been intentionally vague what it means to "compute" a matrix with $\mathbb{Z}_p$ entries, since $\mathbb{Z}_p$ is uncountable. However, since $\mathbb{Z}_p(i)(\mathcal{O}_K/\varpi^n)$ is torsion, the following lemma makes this effective:

Lemma 4.4. *Let (C_*, d) be a chain complex of finitely generated free $\mathbb{Z}_p$-modules, and assume d' is another differential on C_* which differs from d by a multiple of p^n. If $p^{n-1}H_*(C;d')=0$, then $H_*(C;d)\cong H_*(C;d')$.*

In other words: We can only ever know the true differential d up to $O(p^n)$ for large n, but as soon as n exceeds the exponent of the computed cohomology $H_*(C; d')$, it agrees with the true cohomology. So we simply compute with integers up to p-adic precision p^n. Some operations lose precision, most notably application of δ or δ_i, and division by inverses of the isogenies above, but we can bound how much we lose, and as long as we know the precision of the final answer, the lemma tells us whether the result is guaranteed correct. If not, we simply have to restart with a higher initial precision.

5 Even vanishing

In this final talk, we want to study the description

$$\mathbb{Z}_p(i)(\mathcal{O}_K/\varpi^n) \simeq \operatorname{Tot}\begin{pmatrix} \mathrm{F}^{[1,in-1]}\mathrm{N}^{\geq i}\Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} & \longrightarrow & \mathrm{F}^{[0,ni-2]}\mathrm{N}^{\geq i-1}\Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\}\nabla z \\ \big\downarrow \mathrm{can}-\varphi & & \big\downarrow \mathrm{can}-\varphi \\ \mathrm{F}^{[1,in-1]}\Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} & \longrightarrow & \mathrm{F}^{[0,ni-2]}\mathrm{N}^{\geq i-1}\Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\}\nabla z \end{pmatrix}$$

more qualitatively, and draw some general consequences. We start with an easy one:

Theorem 5.1. *$\mathbb{Z}_p(i)(\mathcal{O}_K/\varpi^n)$ is torsion concentrated in degrees -1 and -2, and we have*

$$\frac{|H^1(\mathbb{Z}_p(i)(\mathcal{O}_K/\varpi^n))|}{|H^2(\mathbb{Z}_p(i)(\mathcal{O}_K/\varpi^n)|} = q^{(n-1)i},$$

where q is the order of the residue field of K.

Proof. Since the filtration is bounded, we may pass to the associated graded, on which φ is zero (since it dilates the filtration by a factor of p). So the vertical maps on the F-graded are simply can, and since each of the vertical complexes is already torsion (by the isogeny property of can), we simply ask for the quotients of these "finite Euler characteristics" of the columns. In the below picture, we see $\mathrm{F}^{[1,in-1]}\mathrm{N}^{[0,i-1]}\Delta^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\}$ in blue, and the corresponding part of the ∇ term in red:

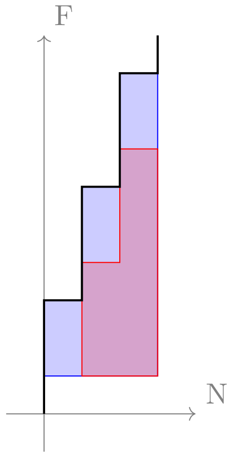

Figure 5: The blue part represents coker(can $-\varphi$) on the left column of the square, the red part represents coker(can $-\varphi$) on the right column of the square. In the picture, $n = 3$ and $i = 3$.

Each square accounts for a copy of k, and the number we look for is q to the power of the difference of blue and red squares. □

Note that this also implies

$$\frac{|K_{2i-1}(\mathcal{O}_K/\varpi^n)|}{|K_{2i-2}(\mathcal{O}_K/\varpi^n)|} = q^{(n-1)i}(q^i - 1)$$

by the comparison between $K(R;\mathbb{Z}_p)$ and $\mathrm{TC}(R)$, the degeneration of the motivic spectral sequence for $\mathrm{TC}(\mathcal{O}_K/\varpi^n)$ for degree reasons, and the fact that $K(\mathcal{O}_K)[\frac{1}{p}] = K(k)[\frac{1}{p}]$.

We observe that the odd groups definitely are always bigger than the even groups. In the crystalline case $R = \mathbb{F}_q[z]/z^n$, computations by Hesselholt-Madsen (and later Sulyma) show that the even groups all vanish in that case. This is equivalent to the maps to the bottom right corner in our square being jointly surjective. However, this is not seen by the F-associated graded alone: can is clearly not surjective, as we just drew its cokernel, the red staircase above. The image of ∇ is controlled by the fact that, on the F-graded, $z^k(s^i \otimes 1) \mapsto kz^{k-1}(s^i \otimes 1)\nabla z$, so in rows which are a multiple of p (or n, in fact, due to the staircase pattern), ∇ also hits only p-multiples. So the F-associated graded has lots of even stuff as well.

The even vanishing in the crystalline case must then come from special properties of the Frobenius! Let us factor the Frobenius through the relative one, and get the following diagram:

$$\begin{array}{ccccc}
\mathrm{N}^{\geq i}\mathbb{\Delta}^{(1)}_R\{i\} & \qquad & \mathrm{N}^{\geq i}\mathbb{\Delta}^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} & \xrightarrow{\nabla} & \mathrm{N}^{\geq i-1}\mathbb{\Delta}^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\}\cdot\nabla z \\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi_{R/\mathbb{Z}_p[[z]]}} & & \downarrow{\scriptstyle\varphi^{\nabla}_{R/\mathbb{Z}_p[[z]]}} \\
\mathbb{\Delta}_R\{i\} & & \mathbb{\Delta}_{R/\mathbb{Z}_p[[z]]}\{i\} & \xrightarrow{\nabla} & \mathbb{\Delta}_{R/\mathbb{Z}_p[[z]]}\{i\}\cdot\nabla z \\
\downarrow{\scriptstyle =} & & \downarrow & & \downarrow \\
\mathbb{\Delta}^{(1)}_R\{i\} & & \mathbb{\Delta}^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\} & \xrightarrow{\nabla} & \mathbb{\Delta}^{(1)}_{R/\mathbb{Z}_p[[z]]}\{i\}\cdot\nabla z
\end{array}$$

Curiously, the inclusion into the Frobenius twist, while very nontrivial on complexes, induces an equivalence on cokernels of ∇, since they compute the same thing. (This somehow a noncanonical lift of the Cartier isomorphism. For $R = \mathrm{F}_q[z]$, the lower two rows reduce mod p to the de Rham forms with zero differential and the de Rham complex respectively.)

With the bases we chose, we know that the $d^c \prod f_u^{e_u}$ with $c + \sum p^u e_u$ go exactly to the $\prod(\delta^u a)^{e_u}$, a basis for the untwisted terms. So the lower vertical maps form a quasi-iso between the rows, while the upper vertical maps form an actual isomorphism at least in the range $\mathrm{F}^{[1,in]}$. (The upper maps preserve filtrations, the lower ones dilate it by p. This is related to being linear and Frobenius-semilinear respectively.)

So we get the following picture:

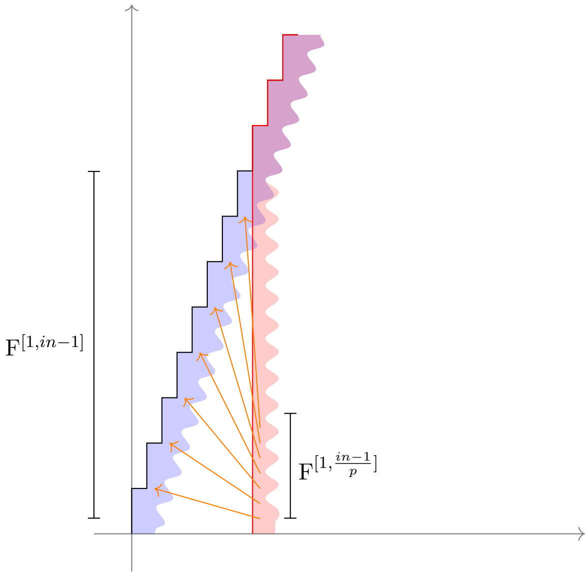

Figure 6: Qualitative picture of the map induced by φ between the cokernels of the horizontal maps in the square. The red shaded area represents the cokernel of ∇ on $N^{\geq i}\mathbb{\Delta}^{(1)}$, the blue shaded area the cokernel of ∇ on $\mathbb{\Delta}^{(1)}$. The Frobenius is an isomorphism between the indicated ranges.

Here the Frobenius takes the $F^{[1,\frac{in-1}{p}]}$ part of the cokernel of ∇ isomorphically to the $F^{[1,in-1]}$ part. In the crystalline situation $R = \mathbb{F}_q[z]/z^n$, the filtration is actually a grading, can lies in strictly lower filtration, and so can $-\varphi$ is surjective between the cokernels of ∇ (essentially since we can induct over the *opposite* filtration, making φ the dominant term). In our case of $R = \mathcal{O}_K/\varpi^n$, this is not possible, since can can also contribute to higher filtration, and so can $-\varphi$ could somehow cancel.

Now the idea is that the $F^{[1,\frac{in-1}{p}]}N^{\geq i}$ region is generated by large powers of d, of exponent roughly $\frac{p-1}{p}i$. As we've seen in Ben's talk, large powers of d actually become on the nose divisible by p at some point. So for large enough i, can is divisible by p on generators of $F^{[1,\frac{in-1}{p}]}N^{\geq i}$. Since φ was a surjection onto the cokernel of ∇ when restricted to those generators, and everything is p-complete, this means that can $-\varphi$ is still surjective onto the cokernel of ∇ and we get:

Theorem 5.2 (Even vanishing)**.** *For $i >> 0$, we have*

$$K_{2i-2}(\mathcal{O}_K/\varpi^n) = 0$$

, and $|K_{2i-1}(\mathcal{O}_K/\varpi^n)| = q^{(n-1)i}$.