

Special values of Motivic L-functions II

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 - ▶ The example of number fields
 - ▶ Determinant functors
- ▶ Talk 2
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Motives and motivic structures (over \mathbb{Q})

$X \rightarrow \text{Spec}(\mathbb{Q})$ smooth, projective variety

$$M_{gm}(X)^* =: h(X) \stackrel{?}{=} \bigoplus_{i=0}^{2 \dim(X)} h^i(X)[-i] \in \text{Ob } DM_{gm}(\mathbb{Q})_{\mathbb{Q}} \quad (\text{def. by Voevodsky})$$

$M = h^i(X)(j)$ for $i, j \in \mathbb{Z}$, more generally $M \in DM_{gm}(\mathbb{Q})_{\mathbb{Q}}^{\heartsuit}$ (heart of conjectural t -structure), leads to a

"Motivic structure":

$$M_I = H_{et}^i(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_I)(j) \quad \text{continuous rep'n of } G_{\mathbb{Q}}$$

$$M_B = H^i(X(\mathbb{C}), \mathbb{Q})(j) \quad \text{pure } \mathbb{Q}\text{-Hodge structure over } \mathbb{R}$$

$$M_{dR} = H_{dR}^i(X/\mathbb{Q})(j) \quad \text{filtered } \mathbb{Q}\text{-vector space}$$

Comparison isomorphisms:

$$M_I \cong M_{B, \mathbb{Q}_I}, \quad M_{B, \mathbb{C}} \cong M_{dR, \mathbb{C}}, \quad M_{I, B_{dR}} \cong M_{dR, B_{dR}}$$

Motivic L-functions

$$P_p(T) = \det(1 - \text{Fr}_p^{-1} \cdot T | M_l^{I_p}) \stackrel{?}{\in} \mathbb{Q}[T]$$

$$L(M, s) = \prod_p P_p(p^{-s})^{-1}, \quad \text{Re}(s) \gg 0$$

- ▶ $M = \mathbb{Q}(j)_F := h^0(\text{Spec}(F))(j)$
 $L(\mathbb{Q}(j)_F, s) = \zeta_F(j + s)$ (Dedekind Zeta-Function)
- ▶ $M = h^0(\text{Spec}(\mathbb{Q}(\sqrt{-1}))) (0) = \mathbb{Q}(0) \oplus \mathbb{Q}(\epsilon)$
 $L(\mathbb{Q}(\epsilon), s) = L(\epsilon, s)$ (Dirichlet L-Function)
- ▶ $E : y^2 = x^3 - x$ ($E/\mathbb{Q}(i)$ has CM by $\mathbb{Z}[i]$)
 $L(h^1(E), s) = L(\phi, s)$ (Hecke L-Function for a character ϕ of $\mathbb{Q}(i)$)
Here $\phi((\alpha)) = \alpha$ where $\alpha \equiv 1 \pmod{(1+i)^3}$
- ▶ $E : y^2 + y = x^3 - x$
 $L(h^1(E), s) = L(f, s)$ (f weight 2 cusp form on $X_0(37)$)

Meromorphic continuation

Conjecture: $L(M, s)$ has meromorphic continuation to all $s \in \mathbb{C}$ and satisfies

$$\Lambda(M, s) = \epsilon(M, s)\Lambda(M^*, 1 - s)$$

where

$$\Lambda(M, s) = L_\infty(M, s)L(M, s)$$

$$L_\infty(M, s) = \prod_{p < q} \Gamma_{\mathbb{C}}(s - p)^{h^{p,q}} \prod_p \Gamma_{\mathbb{R}}(s - p)^{h^{p,+}} \Gamma_{\mathbb{R}}(s - p + 1)^{h^{p,-}}$$

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right), \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$$

Known for

- ▶ $h^0(X) = h^0(\text{Spec}(\Gamma(X, \mathcal{O}_X)))$
- ▶ $h^1(X)$ for X/F an elliptic curve over $F = \mathbb{Q}$ or F real quadratic or cubic (holomorphic continuation) or F totally real or CM (meromorphic continuation)
- ▶ $h^1(X)$ for $X : z_0^n + z_1^n + z_2^n = 0$
- ▶ $\text{Sym}^n h^1(E)$ for E/\mathbb{Q} an elliptic curve
- ▶ $h^d(X)$, X Shimura variety of dimension d

Motivic Cohomology

cycle class map: $CH^j(X)_{\mathbb{Q}} \xrightarrow{cl_j} H^{2j}(X(\mathbb{C}), \mathbb{Q}(j))$

$H^0(M) := \text{Hom}_{DM(\mathbb{Q})_{\mathbb{Q}}}(\mathbb{Q}(0), M) = \text{im}(cl_j) = CH^j(X)_{\mathbb{Q}} / \text{hom}$ for $M = h^{2j}(X)(j)$

$H^1(M) := \text{Hom}_{DM(\mathbb{Q})_{\mathbb{Q}}}(\mathbb{Q}(0), M[1]) = \begin{cases} K_{2j-i-1}^{(j)}(X)_{\mathbb{Q}} & \text{for } M = h^i(X)(j) \\ CH^j(X)_{\mathbb{Q}}^0 = \ker(cl_j) & \text{if } 2j - i - 1 = 0 \end{cases}$

$H_f^0(M) := H^0(M)$

$H_f^1(M) := \text{image of } K_{2j-i-1}^{(j)}(\mathfrak{X})_{\mathbb{Q}}$ where \mathfrak{X} is regular, proper over $\text{Spec}(\mathbb{Z})$

$M_{B,\mathbb{C}} \cong M_{dR,\mathbb{C}}$ induces $\alpha_M : M_{B,\mathbb{R}}^+ \rightarrow (M_{dR}/M_{dR}^0)_{\mathbb{R}}$

Conjecture Mot_∞: $\dim_{\mathbb{Q}} H_f^i(M) < \infty$ and there exists an exact sequence

$$0 \rightarrow H_f^0(M)_{\mathbb{R}} \xrightarrow{c} \ker(\alpha_M) \rightarrow H_f^1(M^*(1))_{\mathbb{R}}^* \xrightarrow{h} H_f^1(M)_{\mathbb{R}} \xrightarrow{r} \text{coker}(\alpha_M) \rightarrow H_f^0(M^*(1))_{\mathbb{R}}^* \rightarrow 0$$

c = cycle class map, h = height pairing, and r = Beilinson regulator.

Vanishing order

Taylor expansion at $s = 0$

$$L(M, s) = L^*(M)s^{r(M)} + \dots$$

Aim: Describe $L^*(M) \in \mathbb{R}^\times$ and $r(M) \in \mathbb{Z}$

Conjecture (Van): $r(M) = \dim_{\mathbb{Q}} H_f^1(M^*(1)) - \dim_{\mathbb{Q}} H_f^0(M^*(1))$

Known cases:

- ▶ F number field, $M = h^0(\text{Spec}(F))(j)$, $j \in \mathbb{Z}$,

$$\dim_{\mathbb{Q}} H_f^1(M^*(1)) - \dim_{\mathbb{Q}} H_f^0(M^*(1)) = \begin{cases} \dim_{\mathbb{Q}} K_{1-2j}(\mathcal{O}_F)_{\mathbb{Q}} & j \leq 0 \\ -1 & j = 1 \\ 0 & j \geq 2 \end{cases}$$

- ▶ $M = h^1(E)(1)$, E/\mathbb{Q} elliptic curve, $\text{ord}_{s=1} L(E, s) \leq 1$, individual examples with $\text{ord}_{s=1} L(E, s) = 2, 3$.

Rationality conjecture

Define **Fundamental Line**

$$\begin{aligned}\Xi(M) := & \det_{\mathbb{Q}}(H_f^0(M)) \otimes \det_{\mathbb{Q}}^{-1}(H_f^1(M)) \\ & \otimes \det_{\mathbb{Q}}(H_f^1(M^*(1))^*) \otimes \det_{\mathbb{Q}}^{-1}(H_f^0(M^*(1))^*) \\ & \otimes \det_{\mathbb{Q}}^{-1}(M_B^+) \otimes \det_{\mathbb{Q}}(M_{dR}/M_{dR}^0),\end{aligned}$$

Conjecture (Rat): $\vartheta_{\infty}(L^*(M)^{-1}) \in \Xi(M) \otimes 1$ where

$$\vartheta_{\infty} : \mathbb{R} \cong \Xi(M) \otimes_{\mathbb{Q}} \mathbb{R}$$

is the isomorphism induced by Conjecture **Mot**_∞.

Known cases:

- ▶ F number field, $M = h^0(\text{Spec}(F))(j)$, $j \in \mathbb{Z}$ (Borel)
- ▶ X/F Shimura curve over totally real F , A direct factor of $\text{Jac}(X)$,
 $M = h^1(A)(1)$, $\text{ord}_{s=1} L(A, s) \leq 1$ (Gross-Zagier-Zhang formula)

An example with $\Xi(M) = \mathbb{Q}$

F totally real, $j < 0$ odd, $M = h^0(\text{Spec}(F))(j)$

$$\Xi(M) = \mathbb{Q}$$

since all spaces in the definition of $\Xi(M)$ are zero!

For $F = \mathbb{Q}$

$$\zeta(1-n) = -\frac{B_n}{n} \quad \text{where} \quad \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$$

For F totally real $\zeta_F(j) \in \mathbb{Q}$ for $j \leq 0$ by the **Klingen-Siegel** theorem.

Galois cohomology

Fix prime l . For each prime p define a complex $R\Gamma_f(\mathbb{Q}_p, M_l)$

$$= \begin{cases} M_l^{!p} \xrightarrow{1-\text{Fr}_p} M_l^{!p} & l \neq p \\ D_{\text{cris}}(M_l) \xrightarrow{(1-\text{Fr}_p, \pi)} D_{\text{cris}}(M_l) \oplus D_{\text{dR}}(M_l) / D_{\text{dR}}^0(M_l) & l = p \end{cases}$$

There is a distinguished triangle of \mathbb{Q}_l -vector spaces.

$$R\Gamma_f(\mathbb{Q}_p, M_l) \rightarrow R\Gamma(\mathbb{Q}_p, M_l) \rightarrow R\Gamma_{/f}(\mathbb{Q}_p, M_l).$$

Let S be a finite set of primes containing l , ∞ and primes of bad reduction. There are distinguished triangles

$$\begin{aligned} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l) &\rightarrow R\Gamma(\mathbb{Z}[\frac{1}{S}], M_l) \rightarrow \bigoplus_{p \in S} R\Gamma(\mathbb{Q}_p, M_l) \\ R\Gamma_f(\mathbb{Q}, M_l) &\rightarrow R\Gamma(\mathbb{Z}[\frac{1}{S}], M_l) \rightarrow \bigoplus_{p \in S} R\Gamma_{/f}(\mathbb{Q}_p, M_l) \\ R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l) &\rightarrow R\Gamma_f(\mathbb{Q}, M_l) \rightarrow \bigoplus_{p \in S} R\Gamma_f(\mathbb{Q}_p, M_l) \end{aligned} \quad (1)$$

Galois cohomology and motivic cohomology

Conjecture Mot_l: a) The cycle class map induces an isomorphism

$$H_f^0(M)_{\mathbb{Q}_l} \cong H_f^0(\mathbb{Q}, M_l) \text{ (Tate conjecture).}$$

b) The Chern class maps induce an isomorphism $H_f^1(M)_{\mathbb{Q}_l} \cong H_f^1(\mathbb{Q}, M_l)$ (Bloch-Kato).

Poitou-Tate duality gives an isomorphism

$$H_f^i(\mathbb{Q}, M_l) \cong H_f^{3-i}(\mathbb{Q}, M_l^*(1))^*$$

for all i . Hence Conjecture **Mot_l** computes the cohomology of $R\Gamma_f(\mathbb{Q}, M_l)$ in all degrees.

Integrality Conjecture

The exact triangle (1) and conjecture \mathbf{Mot}_l induce an isomorphism

$$\vartheta_l : \Xi(M) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong \det_{\mathbb{Q}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l)$$

Let $T_l \subset M_l$ be any $G_{\mathbb{Q}}$ -stable \mathbb{Z}_l -lattice.

Conjecture (Int):

$$\mathbb{Z}_l \cdot \vartheta_l \vartheta_{\infty}(L^*(M)^{-1}) = \det_{\mathbb{Z}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l)$$

This conjecture (for all l) determines $L^*(M) \in \mathbb{R}^{\times}$ up to sign. It is independent of the choice of S and T_l .

Known cases:

- ▶ $M = h^0(\mathrm{Spec}(F))(j)$, $j = 0, 1$ (Analytic class number formula)
- ▶ $M = h^0(\mathrm{Spec}(F))(j)$, $j \in \mathbb{Z}$, F/\mathbb{Q} abelian
- ▶ $M = h^0(\mathrm{Spec}(F))(j)$, $j \in \mathbb{Z}$, or F/K abelian, K imaginary quadratic, $l > 3$ split in K (Johnson-Leung)
- ▶ $M = h^1(E)(1)$, $\mathrm{ord}_{s=1} L(E, s) = 0$, $l \notin S$, S finite, E/\mathbb{Q} CM elliptic curve (Rubin), E/\mathbb{Q} semistable elliptic curve (Kato, Skinner-Urban, Wan)

The equivariant refinement

Let A be a finite-dimensional semisimple \mathbb{Q} -algebra, acting on M .

Examples.

- ▶ X abelian variety, $M = h^1(X)$, $A = \text{End}(X)_{\mathbb{Q}}$
- ▶ X variety with action of a finite group G , e.g.
 $X = X' \times_{\text{Spec}(\mathbb{Q})} \text{Spec}(F)$, F/\mathbb{Q} Galois with group G ,
 $M = h^i(X)(j)$, $A = \mathbb{Q}[G]$.

For simplicity assume A **commutative**, so

$$A \cong E_1 \times \cdots \times E_r, \quad (E_i \text{ number fields})$$

Define $L({}_A M, s)$, $\Xi({}_A M)$, ${}_A \vartheta_{\infty}$, ${}_A \vartheta_l$ as above using the determinant functor over A , $A_{\mathbb{R}}$, $A_l := A \otimes \mathbb{Q}_l$.

- ▶ $L({}_A M, s) \in A_{\mathbb{C}} \cong \prod_{\tau} \mathbb{C}$
- ▶ $r({}_A M) \in H^0(\text{Spec}(A_{\mathbb{C}}), \mathbb{Z}) \cong \prod_{\tau} \mathbb{Z}$
- ▶ $L^*({}_A M) \in (A_{\mathbb{R}})^{\times}$

The equivariant refinement, ctd.

$${}_A\vartheta_\infty : A_{\mathbb{R}} \cong \Xi({}_A M) \otimes_{\mathbb{Q}} \mathbb{R}$$

$${}_A\vartheta_I : \Xi({}_A M) \otimes_{\mathbb{Q}} \mathbb{Q}_I \cong \det_{A_I} R\Gamma_{c,\text{et}}(\mathbb{Z}[\frac{1}{S}], M_I)$$

Equivariant Tamagawa number conjecture - ETNC

Van $r({}_A M) = \dim_A H_f^1(M^*(1)) - \dim_A H_f^0(M^*(1))$

Rat ${}_A\vartheta_\infty(L^*({}_A M)^{-1}) \in \Xi({}_A M) \otimes 1$

Int $\mathfrak{A}_I \cdot {}_A\vartheta_I {}_A\vartheta_\infty(L^*({}_A M)^{-1}) = \det_{\mathfrak{A}_I} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_I)$

Here $\mathfrak{A} \subset A$ is a \mathbb{Z} -order so that there is a **projective** $G_{\mathbb{Q}}$ -stable

$\mathfrak{A}_I := \mathfrak{A} \otimes \mathbb{Z}_I$ -lattice $T_I \subseteq V_I$.

Example. F/K Galois with group G , $\mathfrak{A} = \mathbb{Z}[G]$, $M = h^0(\text{Spec}(F))(j)$

Conj. **Int** known if F/\mathbb{Q} abelian for all j . In general **Rat** not even known for $j = 0, 1$! (Stark conjectures)

Non-commutative coefficients

For any ring R

$$\tau_{\leq 1}K(R) \cong \mathcal{P}(R)$$

where $\mathcal{P}(R)$ is a **Picard category** (groupoid with \otimes). Universal Determinant functor

$$\mathcal{D}^{perf}(R) \cong \rightarrow K(R) \rightarrow \tau_{\leq 1}K(R) \cong \mathcal{P}(R)$$

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If R is commutative semilocal then

$$\pi_0\mathcal{P}(R) = K_0(R) = H^0(\mathrm{Spec}(R), \mathbb{Z}); \quad \pi_1\mathcal{P}(R) = K_1(R) = R^\times$$

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$R = A, A \otimes \mathbb{Q}_I, \mathfrak{A} \otimes \mathbb{Z}_I$ semilocal

If A is non-commutative use universal determinant functor.

Proven cases of the weak TNC

One has the following situation:

- ▶ Conjecture **Mot**_∞ reduces to $H_f^1(M)_{\mathbb{R}} \cong H_{\mathcal{D}}^1(M) := \text{coker}(\alpha_M)$.
- ▶ $\dim_{A_{\mathbb{R}}} H_{\mathcal{D}}^1(M) = 1$.
- ▶ There is $\xi \in H_f^1(M)$ with $A_{\mathbb{R}} \cdot r(\xi) = H_{\mathcal{D}}^1(M)$.

Main example. f elliptic modular form of weight $k \geq 2$, $M = M(f)(j)$, $j \leq 0$.

- ▶ Weak form of **Rat** is known (Bloch-Beilinson)
- ▶ **Int** follows from the main conjecture of Kato/Skinner/Urban if one also assumes $A_{\mathbb{Q}_l} \cdot r_l(\xi) = H_f^1(M_l)$ (Gealy).

Dirichlet L-Functions

$$F = F_m := \mathbb{Q}(\zeta_m); \quad M = h^0(\mathrm{Spec}(F_m))(0)$$

$$G = G_m := \mathrm{Gal}(F_m/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$$

$$A = \mathbb{Q}[G_m] \cong \prod_{\chi \in \hat{G}_{rat}} \mathbb{Q}(\chi)$$

$$L({}_A M, s) = (L(\eta, s))_{\eta \in \hat{G}} \in \prod_{\eta \in \hat{G}} \mathbb{C} \cong A_{\mathbb{C}}$$

Note:

$$\zeta_{F_m}(s) = \prod_{\eta \in \hat{G}} L(\eta, s) \in \mathbb{C}$$

$$\begin{aligned} \mathrm{ord}_{s=0} L(\eta, s) &= \begin{cases} 0 & \eta = 1 \text{ or } \eta(-1) = -1 \\ 1 & \eta \neq 1 \text{ and } \eta(-1) = 1 \end{cases} \\ &= \dim_{\mathbb{Q}(\eta)}(\mathcal{O}_F^\times \otimes_{\mathbb{Z}[G]} \mathbb{Q}(\eta)) \end{aligned}$$

Leading Taylor coefficient at $s = 0$

$$L(\eta, 0) = - \sum_{a=1}^{f_\eta} \left(\frac{a}{f_\eta} - \frac{1}{2} \right) \eta(a)$$

$$\frac{d}{ds} L(\eta, s) \Big|_{s=0} = - \frac{1}{2} \sum_{a=1}^{f_\eta} \log |1 - e^{2\pi i a / f_\eta}| \eta(a)$$

$$\begin{aligned} & \Xi({}_A M)^\# \\ &= \prod_{\substack{\chi \neq 1 \\ \text{even}}} (\mathcal{O}_{F_m}^\times \otimes_{\mathfrak{A}} \mathbb{Q}(\chi))^{-1} \otimes_{\mathbb{Q}(\chi)} (X_{\{v|\infty\}} \otimes_{\mathfrak{A}} \mathbb{Q}(\chi)) \\ & \quad \times \prod_{\text{other } \chi} \mathbb{Q}(\chi) \end{aligned}$$

$${}_A \vartheta_\infty (L^*({}_A M)^{-1})_\chi =$$

$$\begin{cases} 2 \cdot [F_m : F_{f_\chi}] [1 - \zeta_{f_\chi}]^{-1} \otimes \sigma_m & \chi \neq 1 \text{ even} \\ (L(\chi, 0)^\#)^{-1} & \text{else.} \end{cases}$$

Iwasawa-Theory

Let l be a prime, $m \geq 1$

$$\begin{aligned} \Xi_{(A M)}^\# \otimes \mathbb{Q}_l &\xrightarrow{A^{\vartheta_l}} \det_{A_l} R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], M_l)^\# \\ &\cong \det_{A_l} \Delta(F_m) \otimes \mathbb{Q}_l \end{aligned}$$

$$\Delta(F_m) := R\mathrm{Hom}_{\mathbb{Z}_l}(R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], T_l), \mathbb{Z}_l)[-3]$$

Iwasawa-algebra

$$\Lambda = \varprojlim_n \mathbb{Z}_l[G_{m^n}] \cong \mathbb{Z}_l[G_{\ell m_0}][[T]]$$

where

$$m = m_0 l^{\mathrm{ord}_l(m)}; \quad \ell = \begin{cases} l & l \neq 2 \\ 4 & l = 2. \end{cases}$$

Define perfect complex of Λ -modules

$$\Delta^\infty = \varprojlim_n \Delta(L_{m_0 l^n})$$

Iwasawa-Theory, ctd.

Define Elements

$$\eta_{m_0} := (1 - \zeta_{\ell m_0 l^n})_{n \geq 0} \in \varprojlim_n \mathcal{O}_{F_{m_0 l^n}} \left[\frac{1}{ml} \right]^\times \otimes_{\mathbb{Z}} \mathbb{Z}_l = H^1(\Delta^\infty)$$

$$\sigma := (\sigma_{\ell m_0 l^n})_{n \geq 0} \in H^2(\Delta^\infty)$$

$$\theta_{m_0} := (g_{\ell m_0 l^n})_{n \geq 0} \in (\gamma - \chi_{\text{cyclo}}(\gamma))^{-1} \Lambda \subset Q(\Lambda)$$

where

$$g_k = - \sum_{0 < a < k, (a,k)=1} \left(\frac{a}{k} - \frac{1}{2} \right) \tau_{a,k}^{-1} \in \mathbb{Q}[G_k]$$

and $\tau_{a,k} \in G_k$ is defined by $\tau_{a,k}(\zeta_k) = \zeta_k^a$.

$$0 \rightarrow P^\infty \rightarrow H^2(\Delta) \rightarrow X^\infty \rightarrow 0$$

$$P^\infty = \varprojlim_n \text{Pic}(\mathcal{O}_{F_{m_0 l^n}}[1/ml]) \otimes_{\mathbb{Z}} \mathbb{Z}_l, \quad X^\infty = \varprojlim_n X_{\{v | l m_0 \infty\}}(F_{m_0 l^n}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$$

Iwasawa-Theory, ctd.

Total quotient ring

$$Q(\Lambda) \cong \prod_{\psi \in \hat{G}_{\ell m_0}^{\mathbb{Q}_\ell}} Q(\psi)$$

ℓ -adic L-Functions

$$\mathcal{L} := \theta_{m_0}^{-1} + 2 \cdot \eta_{m_0}^{-1} \otimes \sigma \in \det_{Q(\Lambda)} (\Delta^\infty \otimes_\Lambda Q(\Lambda))$$

Theorem(Main Conjecture). One has an equality of invertible Λ -submodules

$$\Lambda \cdot \mathcal{L} = \det_\Lambda \Delta^\infty$$

of $\det_{Q(\Lambda)} (\Delta^\infty \otimes_\Lambda Q(\Lambda))$.

Iwasawa-Theory, ctd.

Since Λ is Cohen-Macaulay (even complete intersection) it suffices to show

$$\Lambda_{\mathfrak{q}} \cdot \mathcal{L} = \det_{\Lambda_{\mathfrak{q}}} \Delta_{\mathfrak{q}}^{\infty}$$

for all **height one** prime ideals \mathfrak{q} .

If $l \notin \mathfrak{q}$ then $\Lambda_{\mathfrak{q}}$ is a d.v.r. with fraction field $Q(\psi_{\mathfrak{q}})$. Main conjecture reduces to

$$\text{Fit}_{\mathfrak{q}}(P_{\mathfrak{q}}^{\infty}) \sim \text{Fit}_{\mathfrak{q}}(H^1(\Delta)_{\mathfrak{q}}/\Lambda_{\mathfrak{q}} \cdot \eta_{m_0}) \quad \psi_{\mathfrak{q}} \text{ even}$$

$$\text{Fit}_{\mathfrak{q}}(P_{\mathfrak{q}}^{\infty}) \sim \theta_{m_0} \quad \psi_{\mathfrak{q}} \text{ odd}$$

which is the classical Iwasawa main conjecture (Theorem of Mazur-Wiles)

For **odd** $l \in \mathfrak{q}$ main conjecture follows from $\mu = 0$ (Ferrero-Washington)

Proof for $l = 2$

For $l = 2$ and \mathfrak{q} a prime ideal of height one of Λ with $2 \in \mathfrak{q}$ the $\Lambda_{\mathfrak{q}}$ -module

$$H^1(\Delta^\infty)_{\mathfrak{q}} \cong H^2(\Delta^\infty)_{\mathfrak{q}} \cong \Lambda_{\mathfrak{q}}/(c - 1)$$

is **not** of finite projective dimension ($c \in \Lambda$ complex conjugation). The determinant $\det_{\Lambda_{\mathfrak{q}}} \Delta_{\mathfrak{q}}^\infty$ **cannot** be computed by passing to cohomology. One needs to construct $\Delta_{\mathfrak{q}}^\infty$ explicitly, using results of Coleman, Leopoldt et al. The proof of the main conjecture for such \mathfrak{q} reduces to a "mod 2 congruence" between

$$(\gamma - \chi_{\text{cyclo}}(\gamma))g_m$$

und

$$(1 - \zeta_m)^{\gamma - \chi_{\text{cyclo}}(\gamma)}$$

expressed by the following Lemma.

Lemma Let $M \equiv 1 \pmod{4}$ be an integer and $0 < x < 1$. The sign of the real number

$$\frac{1 - e^{2\pi ixM}}{(1 - e^{2\pi ix})^M}$$

is $(-1)^{\lfloor xM \rfloor}$.

m_0 odd, $M = 1 + 4m_0 = \chi_{\text{cyclo}}(\gamma)$

$$(1 - \zeta_m^a)^{\gamma - \chi_{\text{cyclo}}(\gamma)} = \frac{1 - e^{2\pi i \frac{a}{m} M}}{(1 - e^{2\pi i \frac{a}{m}})^M}$$

$$(\gamma - \chi_{\text{cyclo}}(\gamma))g_m \equiv \sum_{\substack{0 < a < m \\ (a, m) = 1}} \left\lfloor \frac{aM}{m} \right\rfloor \tau_{a, m}^{-1} \pmod{2}$$

Descent to F_m

For $n \in \mathbb{Z}$ there is a homomorphism $\kappa^n : \Lambda \rightarrow \mathbb{Z}[G_m]$ and an isomorphism

$$\Delta^\infty \otimes_{\Lambda, \kappa^n}^L \mathbb{Z}_l[G_m] \cong R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], T_l(n))$$

For $n \leq 0$ one can compute the image of \mathcal{L} in

$$\det_{\mathbb{Z}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], M_l(n)) \cong \Xi(A M(n)) \otimes \mathbb{Q}_l$$

in terms of Beilinson-Soule elements in $K_{1-2n}(F_m)$, verifying ETNC. For $n = 0$ one needs theorems of Ferrero-Greenberg and Solomon to handle trivial zeros of \mathcal{L} .

To show ETNC for $n > 0$ one proves compatibility of ETNC with the functional equation.

Elliptic curves over \mathbb{Q}

Theorem

(Skinner-Urban, Kato) f elliptic modular form of weight $k = 2$ and level N , p a prime of good ordinary reduction,

- ▶ $\bar{\rho}_f$ irreducible
- ▶ For some $p \neq q \mid N$ $\bar{\rho}_f$ is ramified at q
 $\text{char}(\text{Sel}^\Sigma(T_f)) \sim \mathcal{L}_{\text{alg}}^\Sigma(f)$

Theorem

(X. Wan, Li Cai, Chao Li, Shuai Zhai) The full BSD formula

$$\frac{L(E, 1)}{\Omega_E} = \frac{\#\text{III}(E/\mathbb{Q})}{\#E(\mathbb{Q})^2} \prod_{\ell \mid N} c_\ell$$

holds for certain infinite families of E/\mathbb{Q} with $L(E, 1) \neq 0$. Example: An infinite family of quadratic twists of

$$46A1 : y^2 + xy = x^3 - x^2 - 10x - 12$$

Adjoint motives of modular forms

Theorem

(Diamond-Flach-Guo) f elliptic modular form of weight $k \geq 2$, level N , coefficients in E , Σ set of primes λ of E such that

- ▶ $\lambda \mid Nk!$ or
- ▶ $\bar{\rho}_f$ restricted to $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$ is not abs. irr.

If $\lambda \notin \Sigma$ the TNC holds for $L(\text{Ad}(f), 0)$ and $L(\text{Ad}(f), 1)$

Proof uses Taylor-Wiles method and $R = T$ theorems.

Theorem

(Tilouine-Urban) Under similar assumptions TNC holds for $L(\text{Ad}(f) \otimes \alpha, 0)$ where α is a Dirichlet character corresponding to a real quadratic field F .

Proof uses $R_F = T_F$ and relations between periods of f and f_F .