

Special values of Motivic L-functions I

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- ▶ Talk 1
 - ▶ Some history
 - ▶ The example of number fields
 - ▶ Determinant functors
 - ▶ Tamagawa numbers of algebraic groups
- ▶ Talk 2
 - ▶ General formulation of the Tamagawa number conjecture (Deligne, Beilinson, Bloch, Kato, Fontaine, Perrin-Riou, ...)
 - ▶ Proofs of known cases: Iwasawa Theory and p -adic L-functions
 - ▶ Detailed proof for Dirichlet L-functions
- ▶ Talk 3
 - ▶ Zeta functions of arithmetic schemes
 - ▶ Weil-étale cohomology over finite fields (Lichtenbaum)
 - ▶ Weil-Arakelov cohomology (Flach, Morin)
 - ▶ The example $\text{Spec}(\mathcal{O}_F)$
- ▶ Talk 4
 - ▶ Compatibility with the Conjecture of Birch and Swinnerton-Dyer
 - ▶ Compatibility with the functional equation

Arithmetic schemes vs. motives over \mathbb{Q}

$$\mathfrak{X} = \text{Spec } \mathbb{Z}[X_1, \dots, X_n]/(f_1, \dots, f_m)$$

Zeta function

$$\zeta(\mathfrak{X}, s) = \prod_{x \in \mathfrak{X} \text{ closed}} \frac{1}{1 - N_x^{-s}}$$

L-functions : "Decompose under correspondences"

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L-functions : "Decompose under correspondences"

$X \rightarrow \text{Spec}(\mathbb{Q})$ smooth, projective

Motive: Direct Summand M of $h(X)(n) = \bigoplus h^i(X)(n)[-i]$

For $M = h^i(X)(n)$ one has realisations

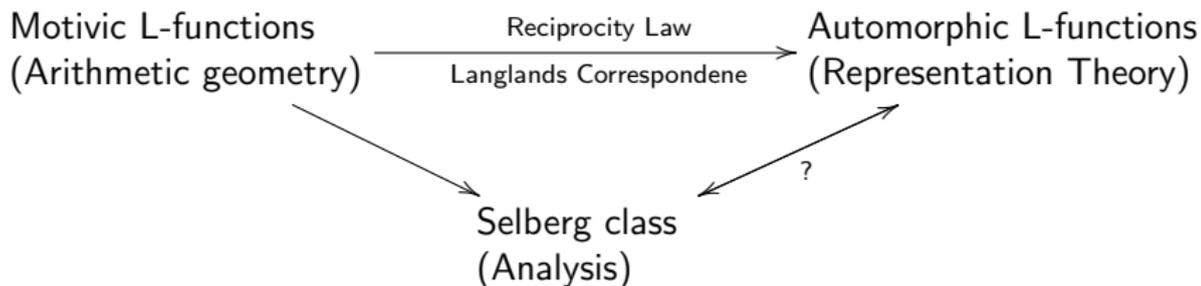
$$M_B = H^i(X(\mathbb{C}), \mathbb{Q}), \quad M_{dR} = H_{dR}^i(X/\mathbb{Q}), \quad M_l = H_{et}^i(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)(n)$$

and motivic cohomology

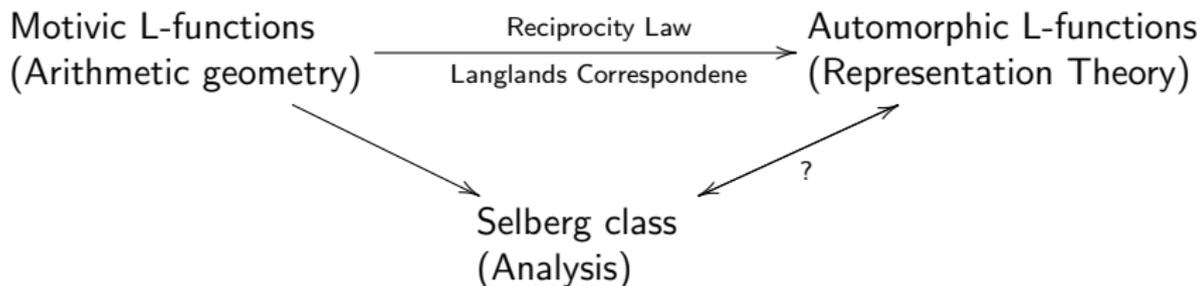
$$H^0(M), H^1(M)$$

and an L-function $L(M, s)$

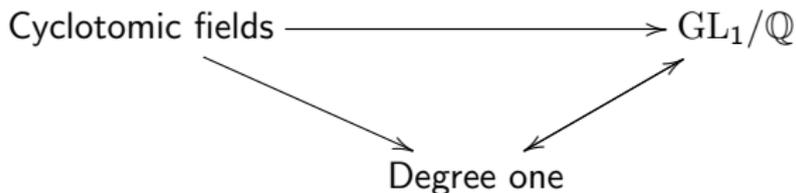
L-functions



L-functions



Basic example: **Dirichlet L-functions**



Leibniz 1673

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

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Euler 1734

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \pi^{2k} \cdot \text{rational number}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k+1}} = ?$$

Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(s) = ?, \quad s = 2, 3, 4, 5, \dots$$

Dirichlet L-function

$$L(\eta, s) = \sum_{n=1}^{\infty} \frac{\eta(n)}{n^s}$$

$$\eta : (\mathbb{Z}/m\mathbb{Z})^\times \rightarrow \mathbb{C}^\times \quad \text{character}$$

example: $m = 4$

$$\epsilon : (\mathbb{Z}/4\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} \{\pm 1\}$$

Generalisation to number fields F

$$\begin{aligned}\zeta_F(s) &= \sum_{\mathfrak{a} \subseteq \mathcal{O}_F} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}} \quad (\text{Dedekind Zeta}) \\ &= \zeta(\text{Spec}(\mathcal{O}_F), s) = L(h^0(\text{Spec}(F)), s)\end{aligned}$$

Analytic class number formula

$$\text{Res}_{s=1} \zeta_F(s) = \frac{2^{r_1} (2\pi)^{r_2} \cdot h \cdot R}{w \cdot \sqrt{|D|}}$$

$$\mathcal{O}_F^\times \xrightarrow{\log | \cdot |_v} \left(\bigoplus_{v|\infty} \mathbb{R} \right)^{\Sigma=0}, \quad R = \text{covol}(\mathcal{O}_F^\times)$$

$$F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$$

$$h = \#\text{Pic}(\mathcal{O}_F), \quad w = \#\mathcal{O}_{F,\text{tors}}^\times, \quad D = \text{disc}(\mathcal{O}_F/\mathbb{Z})$$

Examples

$$F = \mathbb{Q}, r_2 = 0, r_1 = 1$$

$$h = R = D = 1, w = 2$$

$$\operatorname{Res}_{s=1} \zeta(s) = \frac{2^1 \cdot 1 \cdot 1}{2 \cdot 1} = 1$$

Examples

$$F = \mathbb{Q}, \quad r_2 = 0, \quad r_1 = 1$$

$$h = R = D = 1, \quad w = 2$$

$$\operatorname{Res}_{s=1} \zeta(s) = \frac{2^1 \cdot 1 \cdot 1}{2 \cdot 1} = 1$$

$$F = \mathbb{Q}(\zeta_m), \quad G := \operatorname{Gal}(F/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$$

$$\zeta_F(s) = \prod_{\eta \in \hat{G}} L(\eta, s)$$

$$m = 4, \quad \zeta_{\mathbb{Q}(i)}(s) = \zeta(s)L(\epsilon, s)$$

$$\operatorname{Res}_{s=1} \zeta_{\mathbb{Q}(i)}(s) = L(\epsilon, 1) = \frac{2\pi \cdot 1 \cdot 1}{4\sqrt{|-4|}} = \frac{\pi}{4}$$

Generalisation to $s=2,3,4,\dots$

$\text{Res}_{s=1} \zeta_F(s)$ involves $K_0(\mathcal{O}_F) \cong \mathbb{Z} \oplus \text{Pic}(\mathcal{O}_F)$ and $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$

$\zeta_F(m)$ involves $K_{2m-2}(\mathcal{O}_F)$ and $K_{2m-1}(\mathcal{O}_F)$

Theorem (Lichtenbaum, Quillen, Borel, 1973-77)

a) $K_n(\mathcal{O}_F)$ is a finitely generated abelian group

b)

$$K_n(\mathcal{O}_F) \otimes \mathbb{Q} = \begin{cases} 0 & n \equiv 0 \pmod{2} \\ \mathbb{Q}^{r_1+r_2-1} & n = 1 \\ \mathbb{Q}^{r_1+r_2} & n \equiv 1 \pmod{4} \\ \mathbb{Q}^{r_2} & n \equiv 3 \pmod{4} \end{cases}$$

c) $d_m = r_1 + r_2$ (resp. r_2) for $m \equiv 1$ (resp. 0) $\pmod{2}$

$$\zeta_F(m) \sim_{\mathbb{Q}^\times} \frac{\pi^{md-d_m} R_{2m-1}}{\sqrt{|D|}}$$

Computation of $K_n(\mathbb{Z})$

Theorem (Suslin, Voevodsky, Rost et al 1997-2008)

n	0	1	2	3	4	5	6	7
$K_n(\mathbb{Z})$	\mathbb{Z}	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/48$	0	\mathbb{Z}	0	$\mathbb{Z}/240$
8	9	10	11	12	13	14	15	
0?	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/1008$	0?	\mathbb{Z}	0	$\mathbb{Z}/480$	

For $n < 20000$ one knows

n	$8a$	$8a + 1$	$8a + 2$	$8a + 3$
$K_n(\mathbb{Z})$	0?	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2c_{2a+1}$	$\mathbb{Z}/2w_{4a+2}$
	$8a + 4$	$8a + 5$	$8a + 6$	$8a + 7$
	0?	\mathbb{Z}	\mathbb{Z}/c_{2a+2}	\mathbb{Z}/w_{4a+4}

$$c_k = \text{Numerator of } \frac{B_k}{4k}$$

$$w_k = \max\{m \mid k \cdot (\mathbb{Z}/m\mathbb{Z})^\times = 0\}, \quad \text{Example: } w_2 = 24$$

K-Theory vs. (Motivic) Cohomology

$$\zeta(2) = (2\pi)^2 \frac{2}{48} = (2\pi)^2 \frac{\#K_2(\mathbb{Z})}{\#K_3(\mathbb{Z})}$$

$$\zeta(m) \stackrel{?}{=} (2\pi)^{dm-d_m} \frac{\#K_{2m-2}(\mathbb{Z})}{\#K_{2m-1}(\mathbb{Z})_{\text{tor}}} \cdot \frac{R_{2m-1}}{\sqrt{|D|}} \cdot 2^?$$

This is proven using étale cohomology. For primes l and $i = 1, 2$ one has

$$\text{Chern class maps: } K_{2m-i}(\mathbb{Z}) \rightarrow H_{\text{et}}^i(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(m))$$

inducing isomorphisms

$$K_{2m-i}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_l \cong H_{\text{et}}^i(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(m))$$

for $m \geq 2$ and all l if $i = 2$, odd l if $i = 1$. But

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow K_3(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_2 \rightarrow H_{\text{et}}^1(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}_2(2)) \rightarrow 0$$

$$\text{Note: } H_{\text{et}}^1(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}_2(2)) \cong H_{\text{et}}^0(\mathbb{Z}[\frac{1}{2}], \mathbb{Q}_2/\mathbb{Z}_2(2)) \cong \mathbb{Z}/8\mathbb{Z}$$

Tamagawa number conjecture (FPR) for $M = \mathbb{Q}(2)_F$ and F totally real.

Fundamental line

$$\Xi(M) = \det_{\mathbb{Q}} M_{dR}/M_{dR}^0 \otimes \det_{\mathbb{Q}}^{-1} M_B^+ = \det_{\mathbb{Q}} F \otimes \det_{\mathbb{Q}}^{-1} \left(\bigoplus_{v|\infty} (2\pi i)^2 \mathbb{Q} \right)$$

There is a period isomorphism

$$\vartheta_{\infty} : \mathbb{R} \cong \Xi(M) \otimes_{\mathbb{Q}} \mathbb{R}$$

and an isomorphism

$$\vartheta_l : \Xi(M) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong \det_{\mathbb{Q}_l} R\Gamma_c(\mathcal{O}_F[\frac{1}{l}], \mathbb{Q}_l(2))$$

for all primes l . The conjecture says

$$\mathbb{Z}_l \cdot \vartheta_l \vartheta_{\infty}^{-1}(\zeta_F(2)) = \det_{\mathbb{Z}_l} R\Gamma_c(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(2))$$

$$R\Gamma_c(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(2)) \rightarrow R\Gamma(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(2)) \rightarrow R\Gamma(F_{\mathbb{Q}_l}, \mathbb{Z}_l(2)) \oplus R\Gamma(F_{\mathbb{R}}, \mathbb{Z}_l(2))$$

$$M = \mathbb{Q}(2), F = \mathbb{Q} \text{ ctd}$$

$$\vartheta_\infty((2\pi)^2) = \det_{\mathbb{Z}} \mathcal{O}_F \otimes \det_{\mathbb{Z}}^{-1} H^0(\mathbb{R}, (2\pi i)^2 \mathbb{Z})$$

$$\exp : F_{\mathbb{Q}_l} \cong H^1(\mathbb{Q}_l, \mathbb{Q}_l(2))$$

$$\exp(\mathcal{O}_F) = (1 - l^{-2}) H^1(\mathbb{Q}_l, \mathbb{Z}_l(2)) \cdot \#(H^2(\mathbb{Q}_l, \mathbb{Z}_l(2)))^{-1}$$

$$\vartheta_l \text{ also involves } (1 - l^{-2})$$

$$0 \rightarrow H^0(\mathbb{R}, \mathbb{Z}_l(2)) \rightarrow H_c^1(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(2)) \rightarrow H^1(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(2)) \rightarrow$$

$$H^1(\mathbb{Q}_l, \mathbb{Z}_l(2)) \rightarrow H_c^2(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(2)) \rightarrow H^2(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(2)) \rightarrow$$

$$H^2(\mathbb{Q}_l, \mathbb{Z}_l(2)) \oplus H^2(\mathbb{R}, \mathbb{Z}_l(2)) \rightarrow H_c^3(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(2)) \rightarrow 0$$

$$H^2(\mathbb{R}, \mathbb{Z}_l(2)) \cong \mathbb{Z}/2\mathbb{Z} \text{ for } l = 2$$

Insertion: Determinants of perfect complexes

What is a perfect complex?

R ring. A complex C^\bullet of R -Modules is called **perfect** if C^\bullet is quasi-isomorphic to a complex

$$\dots \rightarrow 0 \rightarrow P^n \rightarrow P^{n+1} \rightarrow \dots \rightarrow P^{n+k} \rightarrow 0 \rightarrow \dots$$

where each P^i is a finitely generated projective R -module.

Example: R regular Noetherian, e.g. $R = \mathbb{Z}$,

C^\bullet perfect $\Leftrightarrow H^i(C^\bullet)$ finitely generated and zero for almost all i

R commutative, C^\bullet perfect. Define rank

$$\text{rank}_R C^\bullet := \sum_{i \in \mathbb{Z}} (-1)^i \text{rank}_{\mathbb{Z}} P^i \in H^0(\text{Spec}(R), \mathbb{Z})$$

and the determinant

$$\det_R C^\bullet = \bigotimes_{i \in \mathbb{Z}} \left(\bigwedge^{\text{rank}_R P^i} P^i \right)^{(-1)^i}$$

an invertible R -module.

Determinants of perfect complexes, ctd

Main properties:

A) A short exact sequence of complexes

$$0 \rightarrow C_1^\bullet \rightarrow C_2^\bullet \rightarrow C_3^\bullet \rightarrow 0$$

induces canonical isomorphism

$$\det_R C_2^\bullet \cong \det_R C_1^\bullet \otimes_R \det_R C_3^\bullet$$

B) For acyclic C^\bullet there is a canonical isomorphism

$$\det_R C^\bullet \cong R$$

C) $C_1^\bullet \xrightarrow{\sim} C_2^\bullet$ quasi-isomorphism induces

$$\det_R C_1^\bullet \xrightarrow{\sim} \det_R C_2^\bullet$$

D) For R regular Noetherian there are canonical isomorphisms

$$\det_R C^\bullet \cong \bigotimes_{i \in \mathbb{Z}} \det_R^{(-1)^i} H^i(C^\bullet)$$

Determinants of perfect complexes, ctd

The isomorphism in **A**) depends on the ordering. Take free, rank one R -module V . The diagram

$$\begin{array}{ccccc}
 \det_R(V \oplus V) & \longrightarrow & \det_R V \otimes \det_R V & \xlongequal{\quad} & V \otimes V \\
 \downarrow \text{flip} & & \downarrow \text{flip} & & \downarrow \text{flip} \\
 \det_R(V \oplus V) & \longrightarrow & \det_R V \otimes \det_R V & \xlongequal{\quad} & V \otimes V
 \end{array}$$

does not commute since $v_1 \wedge v_2 = -v_2 \wedge v_1$ but $v_1 \otimes v_2 = v_2 \otimes v_1$. Hence define **graded line bundles** (L, α) where $\alpha \in H^0(\text{Spec}(R), \mathbb{Z})$ with

$$(L, \alpha) \otimes (M, \beta) := (L \otimes M, \alpha + \beta)$$

and commutativity constraint

$$l \otimes m = (-1)^{\alpha\beta} m \otimes l.$$

$\text{Det}_R : \text{Perfect complexes} \rightarrow \text{Graded line bundles}$

$$\text{Det}_R C^\bullet = (\det_R C^\bullet, \text{rank}_R C^\bullet)$$

Determinants of perfect complexes, ctd

If $R = \mathbb{Z}_I$ and M is a finite R -module then

$$\det_{\mathbb{Q}_I}(M \otimes_{\mathbb{Z}_I} \mathbb{Q}_I) = \det_{\mathbb{Q}_I}(0) = \mathbb{Q}_I$$

and

$$\det_{\mathbb{Z}_I}(M) = \mathbb{Z}_I \cdot |M|^{-1} \subset \mathbb{Q}_I$$

- ▶ R a finite free \mathbb{Z}_I -algebra
- ▶ T a finitely generated projective R -module with continuous R -linear action of $\pi_{1,et} \text{Spec}(\mathbb{Z}[\frac{1}{S}])$
- ▶ $R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T)$, $R\Gamma(\mathbb{Q}_I, T)$ are perfect complexes of R -modules, $R\Gamma(\mathbb{Z}[\frac{1}{S}], T)$ is not in general.

The Class number formula from the point of view of TNC

$$\operatorname{Res}_{s=1} \zeta_F(s) = \frac{2^{r_1} (2\pi)^{r_2} \cdot h \cdot R}{w \cdot \sqrt{|D|}}$$

Algebraic group \mathbb{G}_m with tangent space \mathbb{G}_a (over F).

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Ring of Adeles

$$\mathbb{A}_F = \prod'_{\text{all } v} F_v = \varinjlim_{S \text{ finite}} \left(\prod_{v \in S} F_v \times \prod_{v \notin S} \mathcal{O}_{F_v} \right)$$

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The locally compact group $\mathbb{G}_m(\mathbb{A}_F)$ has a canonical measure (Tamagawa measure) because of the product formula for F .

The class number formula is equivalent to

$$\operatorname{vol}(\mathbb{G}_m(\mathbb{A}_F)/\mathbb{G}_m(F)) = \operatorname{vol}(\mathbb{A}_F^\times/F^\times) = 1$$

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Generalizes to Tori (Ono), reductive groups (Tamagawa, Weil,...), abelian varieties (BSD, Tate) and motives (Bloch/Kato)

The Class number formula from the point of view of TNC (FPR)

$$\operatorname{Res}_{s=1} \zeta_F(s) = \frac{2^{r_1} (2\pi)^{r_2} \cdot h \cdot R}{w \cdot \sqrt{|D|}}$$

Motive: $M = \mathbb{Q}(1)_F$

Betti realization: $M_B = \bigoplus_{\tau \in \Sigma} (2\pi i) \mathbb{Q}_\tau$, $\Sigma = \operatorname{Hom}(F, \mathbb{C})$

Fundamental Line: $\Xi(M) :=$

$$\begin{aligned} & \operatorname{Det}_{\mathbb{Q}}^{-1}(H_f^1(M)) \otimes \operatorname{Det}_{\mathbb{Q}}^{-1}(H_f^0(M^*(1))^*) \otimes \operatorname{Det}_{\mathbb{Q}}(M_{dR}/M_{dR}^0) \otimes \operatorname{Det}_{\mathbb{Q}}^{-1}(M_B^+) \\ & \cong \operatorname{Det}_{\mathbb{Q}}^{-1}((\mathcal{O}_F^\times)_{\mathbb{Q}}) \otimes \operatorname{Det}_{\mathbb{Q}}^{-1}(\mathbb{Q}) \otimes \operatorname{Det}_{\mathbb{Q}} F \otimes \operatorname{Det}_{\mathbb{Q}}^{-1} \left(\bigoplus_{\substack{v|\infty \\ v \text{ complex}}} (2\pi i)(\tau_v - \bar{\tau}_v) \mathbb{Q} \right) \end{aligned}$$

$$0 \rightarrow M_{B,\mathbb{R}}^+ \xrightarrow{\alpha_M} \bigoplus_{v|\infty} F_v \cong F_{\mathbb{R}} \rightarrow \operatorname{coker}(\alpha_M) \cong \bigoplus_{v|\infty} \mathbb{R} \rightarrow 0$$

$$0 \rightarrow (\mathcal{O}_F^\times)_{\mathbb{R}} \xrightarrow{\log | \cdot |_v} \operatorname{coker}(\alpha_M) \rightarrow \mathbb{R} \rightarrow 0$$

induces

$$\vartheta_\infty : \mathbb{R} \cong \Xi(M) \otimes_{\mathbb{Q}} \mathbb{R}$$

$$\vartheta_\infty \left(\frac{(2\pi)^{r_2} R}{\sqrt{|D|}} \right) = (\eta_1 \wedge \cdots \wedge \eta_{r_1+r_2-1})^{-1} \otimes (1^*)^{-1} \otimes (\omega_1 \wedge \cdots \wedge \omega_d) \otimes \bigwedge_{v|\infty} (\tau_v - \bar{\tau}_v)^{-1}$$

The Class number formula from the point of view of TNC (FPR), ctd

$\eta_1, \dots, \eta_{r_1+r_2-1}$ \mathbb{Z} -basis of $\mathcal{O}_F^\times / \text{tor}$

$\omega_1, \dots, \omega_d$ \mathbb{Z} -basis of \mathcal{O}_F

1 \mathbb{Z} -basis of $H^0(\text{Spec } F, \mathbb{Z})$

$$R\Gamma_c(\mathcal{O}_F[\frac{1}{I}], \mathbb{Z}_I(1)) \rightarrow R\Gamma_f(F, \mathbb{Z}_I(1)) \rightarrow R\Gamma_f(F_{\mathbb{Q}_I}, \mathbb{Z}_I(1)) \oplus R\Gamma_f(F_{\mathbb{R}}, \mathbb{Z}_I(1))$$

$$0 \rightarrow M_{B, \mathbb{Z}}^+ \otimes \mathbb{Z}_I \cong \bigoplus_{v|\infty} H^0(F_v, \mathbb{Z}_I(1)) \rightarrow H_c^1(\mathcal{O}_F[\frac{1}{I}], \mathbb{Z}_I(1)) \rightarrow \mathcal{O}_F^\times \otimes \mathbb{Z}_I$$

$$\rightarrow \bigoplus_{v|I} \hat{\mathcal{O}}_{F_v}^\times \oplus \bigoplus_{v|\infty} H^1(F_v, \mathbb{Z}_I(1)) \rightarrow H_c^2(\mathcal{O}_F[\frac{1}{I}], \mathbb{Z}_I(1)) \rightarrow \text{Pic}(\mathcal{O}_F) \otimes \mathbb{Z}_I \rightarrow 0$$

$$H_c^2(\mathcal{O}_F[\frac{1}{I}], \mathbb{Z}_I(1)) \cong \mathbb{Z}_I$$

$$H_f^1(F_v, \mathbb{Z}_I(1)) := \hat{\mathcal{O}}_{F_v}^\times \xrightarrow{\subset} F_v^\times \otimes \mathbb{Z}_I \cong H^1(F_v, \mathbb{Z}_I(1))$$

The Class number formula from the point of view of TNC (FPR), ctd

The exponential series $\exp(x) = 1 + x + \frac{x^2}{2} + \dots$ gives an isomorphism

$$\mathcal{O}_{F_v} \supset (I) = \mathfrak{m}_v^e \cong 1 + \mathfrak{m}_v^e \subset \mathcal{O}_{F_v}^\times$$

hence

$$\det_{\mathbb{Z}_I} \mathcal{O}_{F_v} = \frac{Nv - 1}{Nv} \det_{\mathbb{Z}_I} \hat{\mathcal{O}}_{F_v}^\times$$

inside

$$\det_{\mathbb{Q}_I} \hat{\mathcal{O}}_{F_v}^\times \otimes_{\mathbb{Z}_I} \mathbb{Q}_I = \det_{\mathbb{Q}_I} H_f^1(F_v, \mathbb{Q}_I(1)).$$

But

$$\vartheta_I : \Xi(M) \otimes_{\mathbb{Q}} \mathbb{Q}_I \cong \det_{\mathbb{Q}_I} R\Gamma_c(\mathcal{O}_F[\frac{1}{I}], \mathbb{Q}_I(1))$$

also involves a factor $1 - Nv^{-1}$ by definition. Hence

$$\mathbb{Z}_I \cdot \vartheta_I \vartheta_\infty^{-1}(\zeta_F(1)^*) = \text{Det}_{\mathbb{Q}_I} R\Gamma_c(\mathcal{O}_F[\frac{1}{I}], \mathbb{Z}_I(1))$$

Advantages of the Fontaine/Perrin-Riou formulation

- ▶ Generalizes to a description of the leading Taylor-coefficients $\zeta_F^*(s)$ at $s = 0, -1, -2, \dots$
- ▶ Generalizes to Hasse-Weil L-functions $L(M, s)$ of motives M over number fields at any $s \in \mathbb{Z}$
- ▶ Generalizes to motives with coefficients in a semisimple \mathbb{Q} -algebra A .

Time line

Year	Mathematician	Theorem or Conjecture
1673	Leibniz	formula for $\frac{\pi}{4}$
1734	Euler	formula for $\zeta(2k)$
1838	Dirichlet	Analytic Class number formula for quadratic fields
1850	Kummer	Analytic Class number formula for cyclotomic fields
1859	Riemann	Analytic continuation and functional equation of $\zeta(s)$, Riemann hypothesis
1896	Dedekind	Analytic Class number formula for number fields
1917	Hecke	Analytic continuation and functional equation of $\zeta_F(s)$
1921	Artin	Zeta function for global fields of finite characteristic
1934	Hasse	Riemann hypothesis in char. p and $g = 1$
1949	Weil	Weil-Conjectures
1960	Dwork	Rationality of Zeta-Functions in char. p
1963	Ono	Tamagawa numbers of Tori
1965	Birch, Sw.-Dyer	Birch and Swinnerton-Dyer (BSD) conjecture for elliptic curves

Year	Mathematician	Theorem or Conjecture
1963	Grothendieck et al	Cohomological formula for Zeta-Functions in char. p
1966	Tate	BSD-Conjecture for abelian varieties, Tate Conjecture
1969	Serre	Definition of motivic L -Functions
1972	Quillen	Definition of algebraic K -groups
1974	Borel	Relation between $\zeta_F(m)$ and $K_{2m-1}(\mathcal{O}_F)$
1974	Deligne	Riemann hypothesis in char. p
1976	Coates, Wiles	rank-part of BSD for CM elliptic curves of rank 0
1978	Bloch	$K_2(E) \sim L(E, 2)$ for certain elliptic curves
1979	Deligne	rationality conjecture for $L(M, 0)$ for critical motives M
1983	Gross, Zagier	Gross-Zagier formula for $L'(E, 1)$
1984	Mazur, Wiles	Iwasawa main conjecture for cyclotomic fields
1985	Beilinson	rationality conjecture for $L(M, s)$
1986	Rubin	first examples of finite Tate-Shafarevich groups
1988	Bloch, Kato	integrality conjecture for $L(M, s)$ for $w(M) - 2s < 0$
1988	Kolyvagin	Euler systems, rank-part of BSD for $\text{ord}_{s=1} L(E, s) \leq 1$

Year	Mathematician	Theorem or Conjecture
1991	Fontaine, Perrin-Riou	integrality conjecture for $L(M, s)$ for all s , determinants
1999	Burns, Flach	equivariant conjecture with non-commutative coefficients, Galois module theory
2000	Kings	TNC for $L(E, s)$, E CM, $s = 2, 3, 4, \dots$
2003	Huber, Kings	TNC for Dirichlet L-functions
2003	Burns, Greither	ETNC for Dirichlet L-functions
2005	Fukaya, Kato	General Iwasawa main conjecture
2014	Skinner, Urban, Kato	Iwasawa main conjecture for elliptic curves over \mathbb{Q}

Partly based on: Tejaswi Navilarekallu, On the Equivariant Tamagawa Number Conjecture, Thesis, Caltech 2006.