YTM 2022 Cellular Sheaves & Applications

A few exercises, for fun. Robert Ghrist, 2022.

1. Give a short proof of the result that if $h : X \to \mathbb{Z}$ is a constructible function of the form

$$h = \sum_{k=1}^{N} \mathbf{1}_{U_k} \quad : \quad \chi(U_k) = C$$

then N can be computed from h via the normalized integral

$$N = \frac{1}{C} \int_X h \, d\chi$$

2. Draw an example of a constructible function $h : \mathbb{R}^2 \to \mathbb{Z}$ which is strictly positive and yet has vanishing Euler integral. Hmmmm... that's weird.

3. Hey, wait a minute! The result of problem 1 only works if $C \neq 0$; that is, the "footprint" sets are not sets of (Euler) measure zero. But who cares about sets of measure zero, right? Hmmmm... Find an example of a constructible $h : \mathbb{R}^2 \to \mathbb{Z}$ which is a sum of indicator functions on closed annuli, but which can be decomposed into such a sum in multiple ways, with different numbers of annuli. This means that the theorem in problem 1 is really an Euler-calculus result, and not due to some more primal principle.

4. Let's say you have cellular sheaves $\mathcal{F}$ and $\mathcal{G}$ taking values in a fixed data category D over cell complexes X and Y respectively. Given a cellular map $\varphi : X \to Y$ (a map which takes cells to cells respecting the face relation), what do you think are the appropriate definitions for:

   1. The direct image sheaf $\varphi_* \mathcal{F}$ on Y;

   2. The inverse image sheaf $\varphi^* \mathcal{G}$ on X.

Try it out on a few simple examples.

5. In discussions about politics/opinions/preferences, pairwise agreement between individuals does not always mean that the entire group will agree: there are some weird dynamics that kick in with larger groups. Use cohomology to argue that the discourse sheaf model we have discussed *fails* to capture that possibility. That is, if you try to extend a discourse sheaf over a graph to a more general cellular sheaf over a complex (with higher dimensional simplices), then the group-agreement is forced by pairwise-agreement.

6. Assume a simple discourse sheaf over a graph where each stalk is 1-dimensional (everyone has an opinion about and discusses The Current Thing). Is it possible to determine whether someone is lying, simply based on observing the image of the coboundary operator on a 0-cochain? (So, you can't observe either the 0-cochain; only its coboundary)