CELLULAR SHEAVES, COHOMOLOGY, & APPLICATIONS

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PART 3

OUTLINE

REASONING WITH SHEAVES

NONABELIAN GOALS

HARMONIC CONVERGENCE

SIMPLE EXAMPLES

CELLULAR SHEAVES

PROLOGUE : CALCULUS

BEYOND (VECTOR) SPACE

CO/SHEAVES TAKING VALUES IN...

THERE ARE SEVERAL APPLICATIONS FOR SHEAVES WITH MORE GENERAL DATA CATEGORIES

SETS

- REEB GRAPHS, SMOOTHING, APPROXIMATION — DE SILVA, MUNCH, & PATEL
- CONTEXTUALITY, PARADOX — ABRAMSKY + AL.

SEMIGROUPS

- POSITIVE CO/HOMOLOGY FOR SHEAVES OVER TIME AXIS — G + KRISHNAN

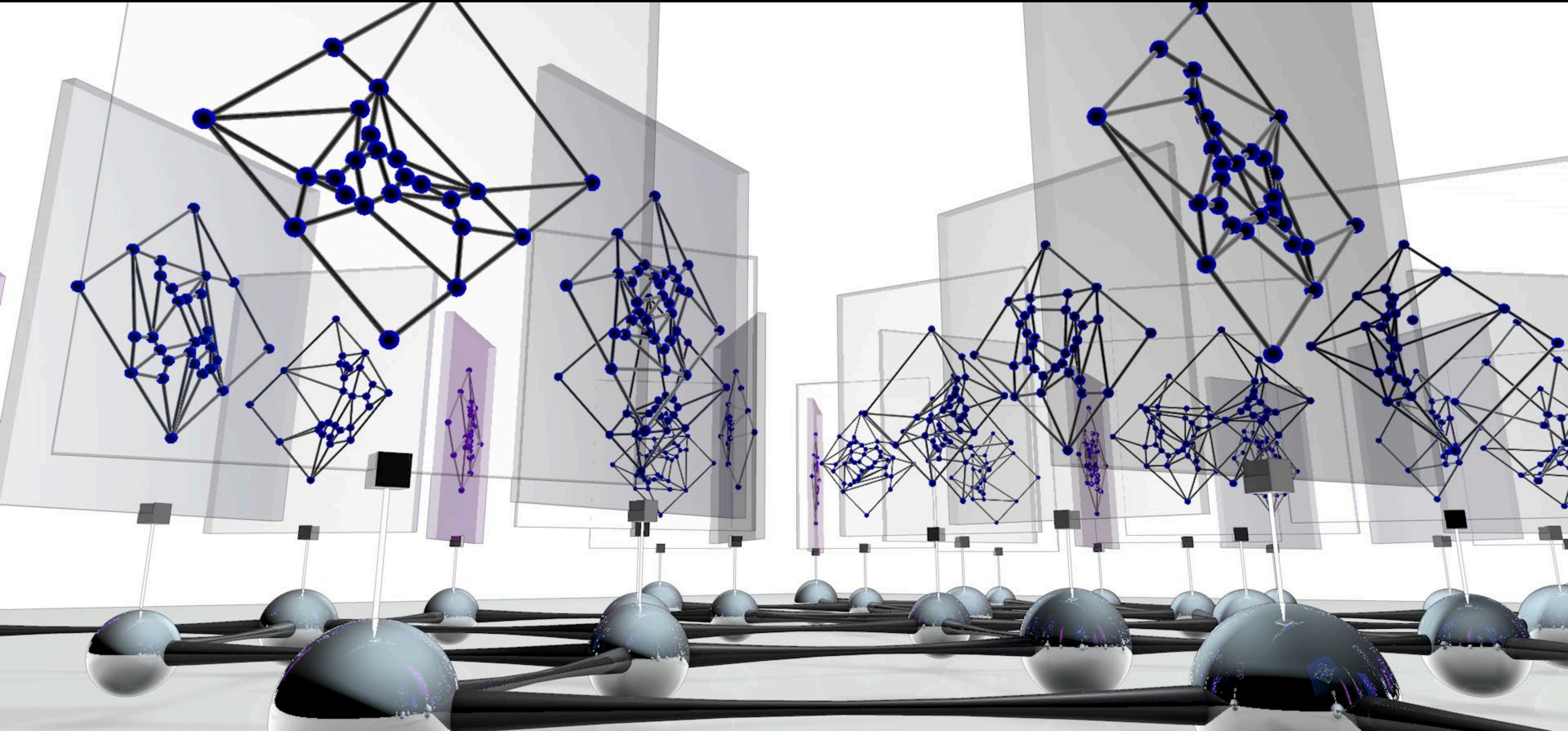
$\mathbb{Z}$-MODULES

- GENERALIZED PERSISTENCE ; BI-SHEAVES ; VERY GENERAL! — PATEL, MACPHERSON-PATEL

SOMETHING MORE GENERAL...

BASED ON WORK WITH HANS RIESS

SHEAVES OF LATTICES



BACKGROUND : LATTICES

LATTICES

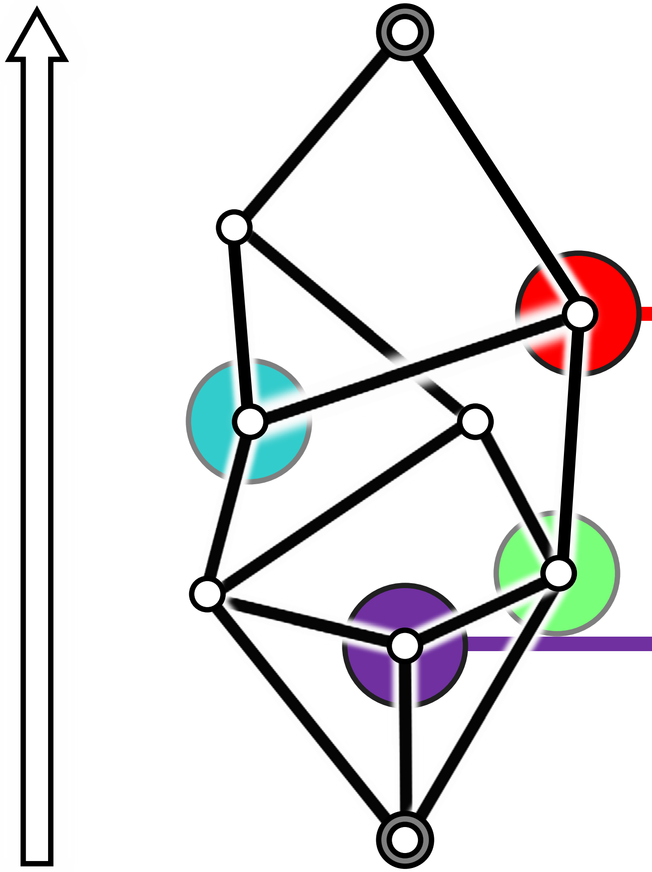
LATTICES are PARTIALLY ORDERED SETS with a PAIR of OPERATIONS

V : Λ

COMMUTATIVE

ASSOCIATATIVE

IDEMPOTENT



V : JOIN

LEAST UPPER BOUND

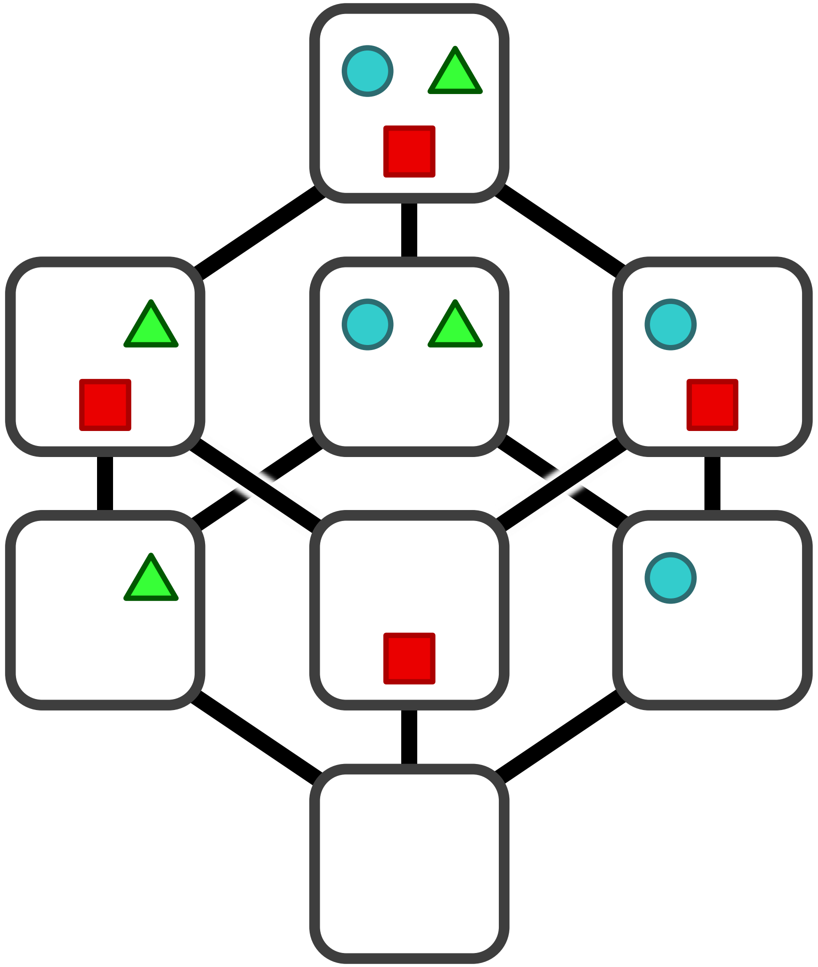
UNION ; OR ; MAX ...

Λ : MEET

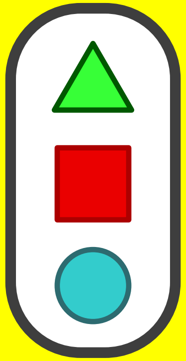
GREATEST LOWER BOUND

INTERSECTION ; AND ; MIN ...

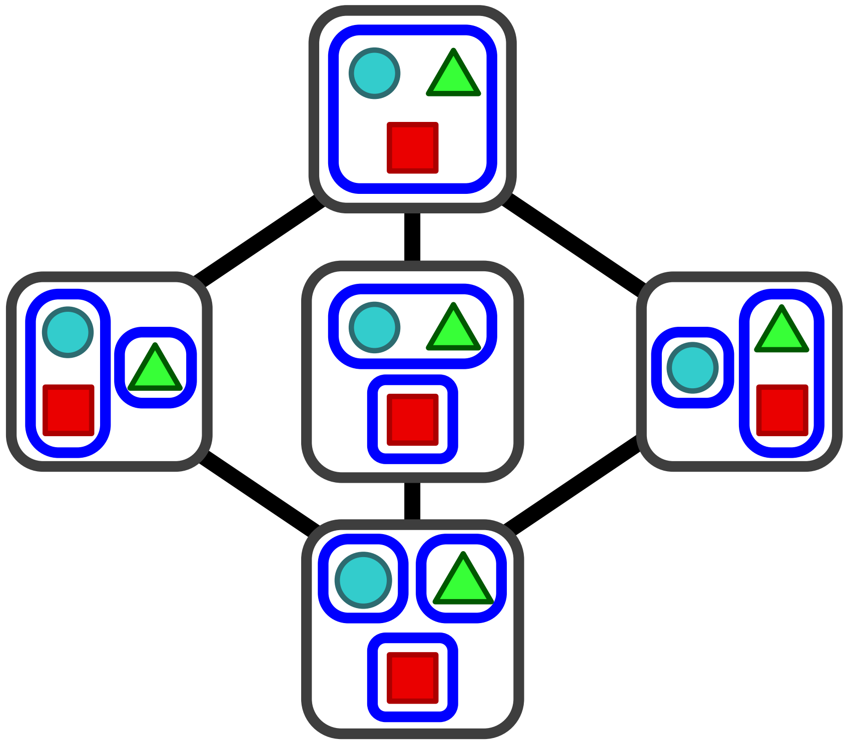
EXAMPLES OF LATTICES



2^X = POWER SET of X



X



$P(X)$ = PARTITIONS of X

GALOIS / CONCEPT LATTICES

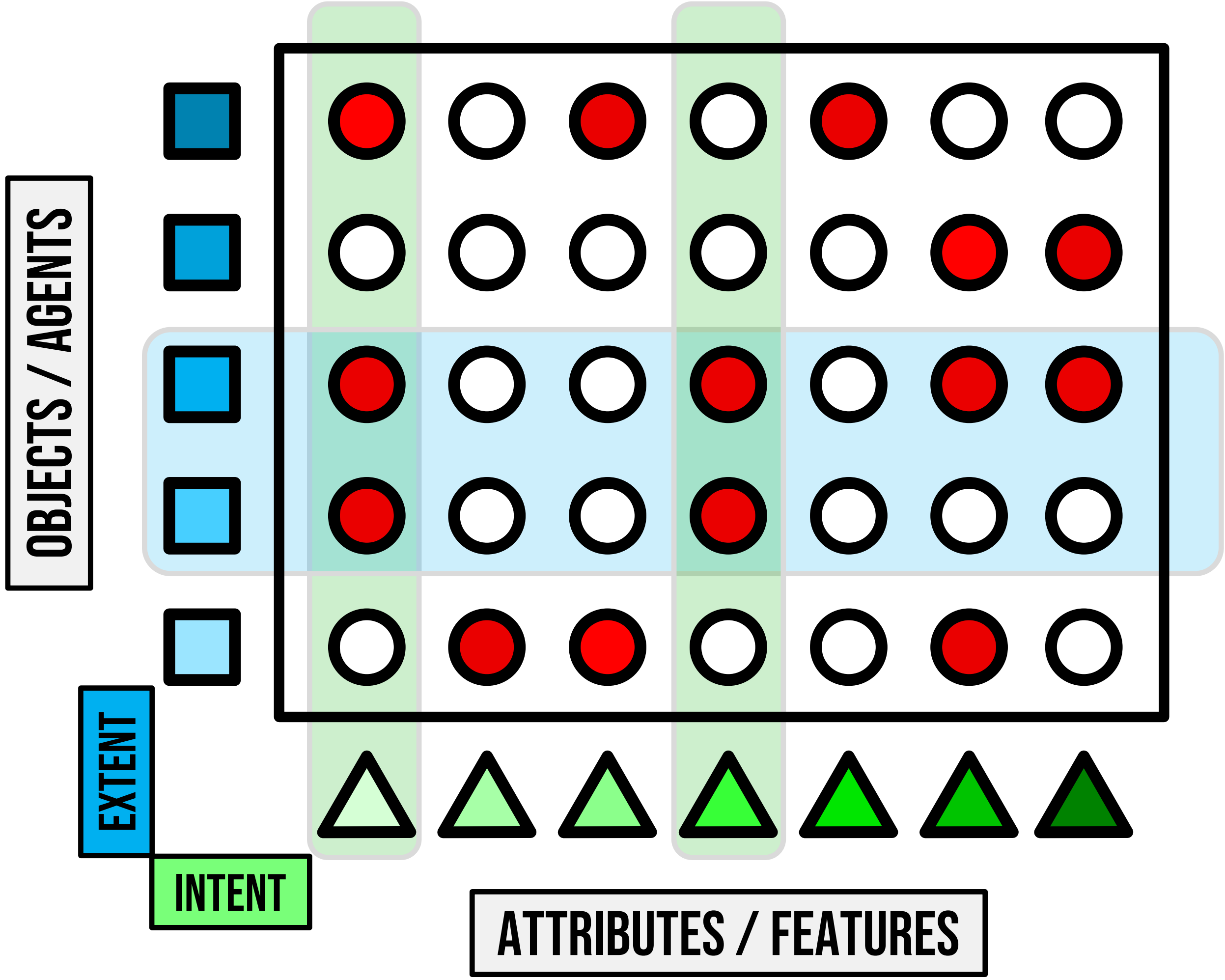
Given a binary relation on two sets:

Acts as a **KERNEL** for a transform...

Inverse blocks are called **CONCEPTS**

Concepts are pairs **INTENT-EXTENT**

Concepts generate an ordering/lattice



CONCEPT LATTICE
GALOIS LATTICE
[Wille, 1981 + more...]

GALOIS / CONCEPT LATTICES

EXAMPLE : Attributes of various social media platforms

		TEXT	IMAGE	VIDEO	DIRECTED	FOLLOWS	UP/LIKE	DOWN	USA	PRIVATE	DISINFO
		TXT	IMAG	VID	DIR	FOLL	UP	DOWN	USA	PRIV	DIS
TWITTER	Tw	■	■	■	■	■	■		■	■	■
FACEBOOK	Fb	■	■	■		■	■		■		■
INSTAGRAM	Ig	■	■	■	■	■	■		■	■	■
TIKTOK	Tk	■		■	■	■	■			■	■
DISCORD	Di	■	■						■		■
YOUTUBE	Yt	■		■	■	■	■	■	■	■	■
REDDIT	Re	■	■		■		■	■	■		■

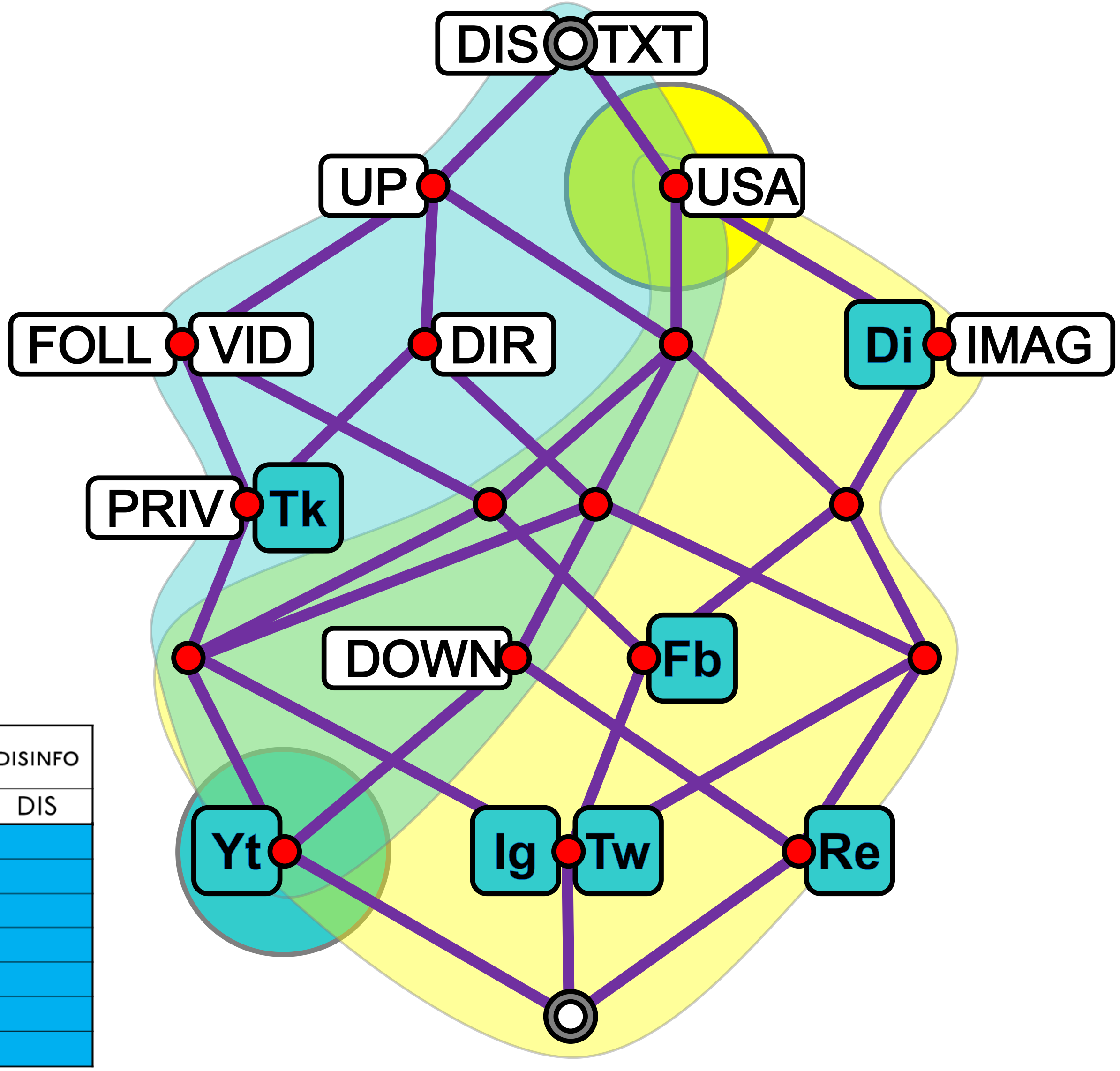
GALOIS / CONCEPT LATTICES

The GALOIS LATTICE arranges the object-attribute pairs

It is illustrative to examine

UPSETS : Yt↑

DOWNSETS : USA↓



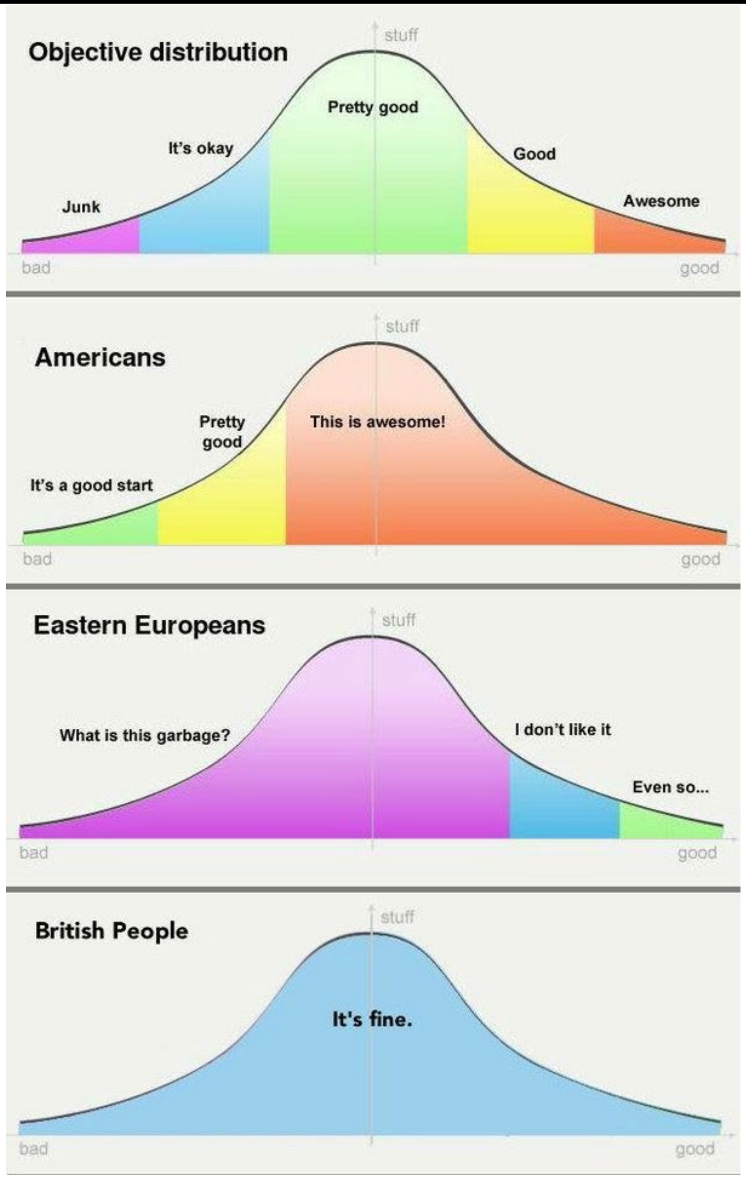
		TEXT	IMAGE	VIDEO	DIRECTED	FOLLOWS	UP/LIKE	DOWN	USA	PRIVATE	DISINFO
		TXT	IMAG	VID	DIR	FOLL	UP	DOWN	USA	PRIV	DIS
TWITTER	Tw	■	■	■	■	■	■		■	■	■
FACEBOOK	Fb	■	■	■		■	■		■		■
INSTAGRAM	Ig	■	■	■	■	■	■		■	■	■
TIKTOK	Tk	■		■	■	■	■			■	■
DISCORD	Di	■	■						■		■
YOUTUBE	Yt	■		■	■	■	■	■	■	■	■
REDDIT	Re	■	■		■		■	■	■		■

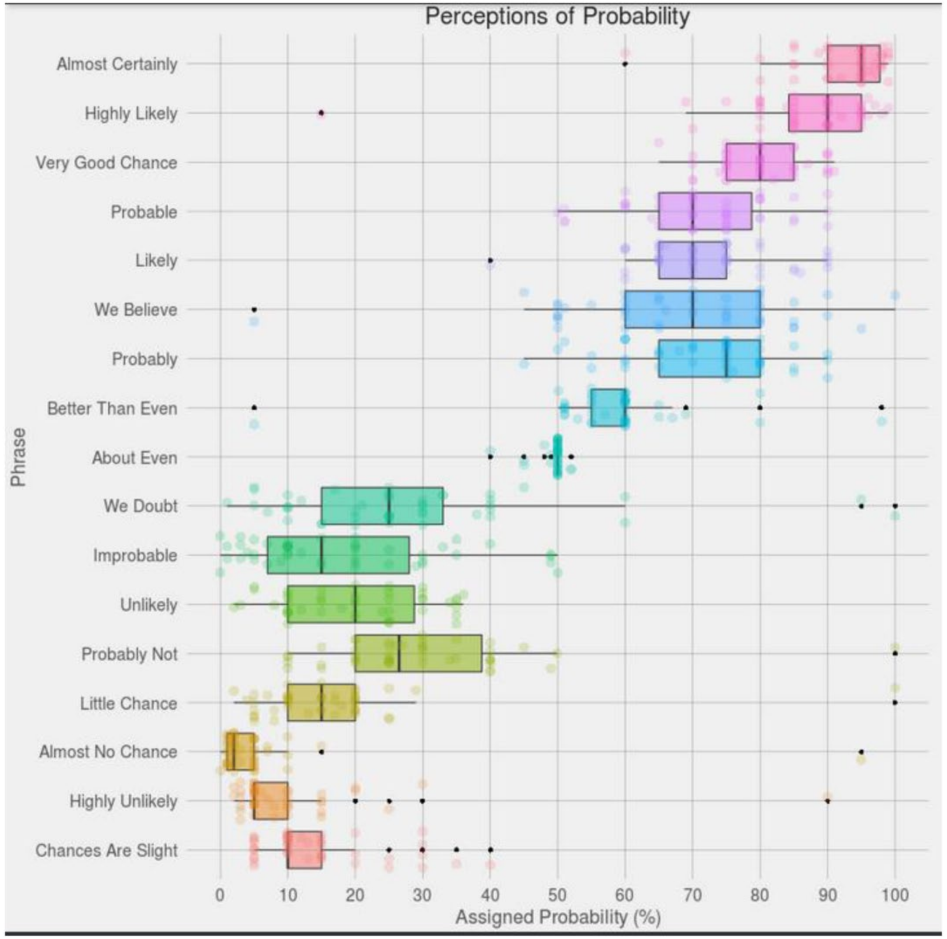
IDEA : NONLINEARITY VIA LATTICES

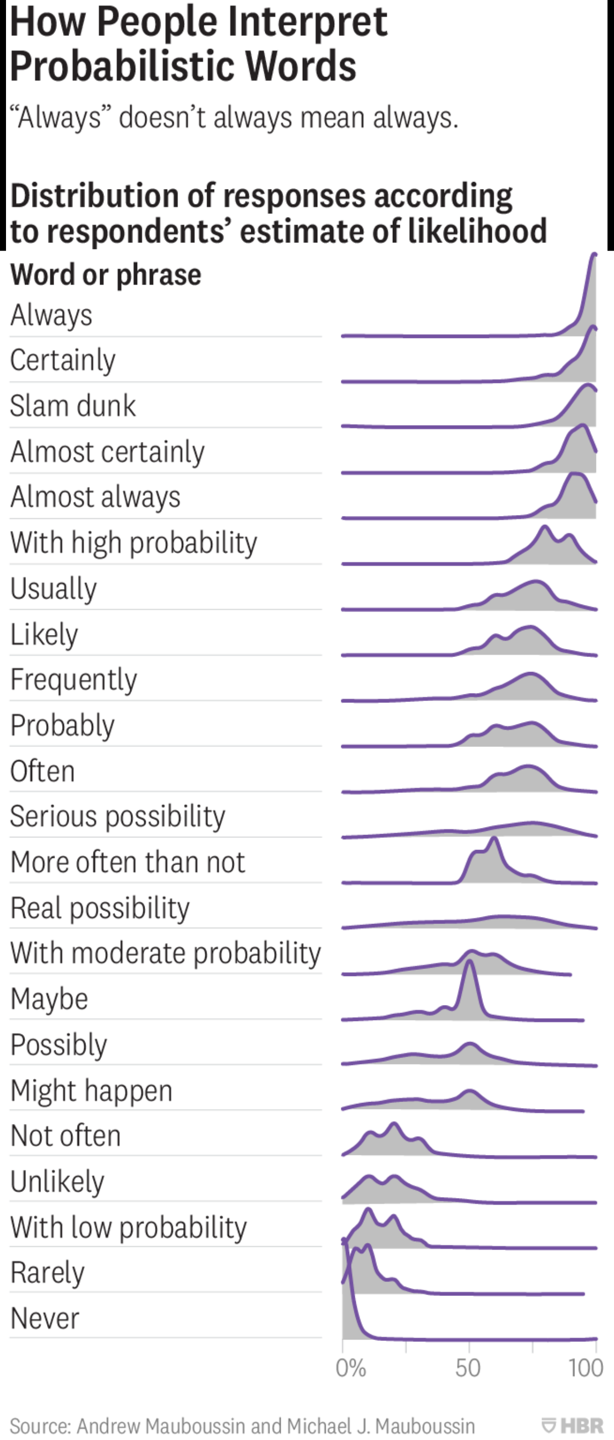
IF YOU WANT NONLINEAR SHEAVES, NON-LINEAR ORDERED SETS IS A GOOD START…

PERSONAL SCALES

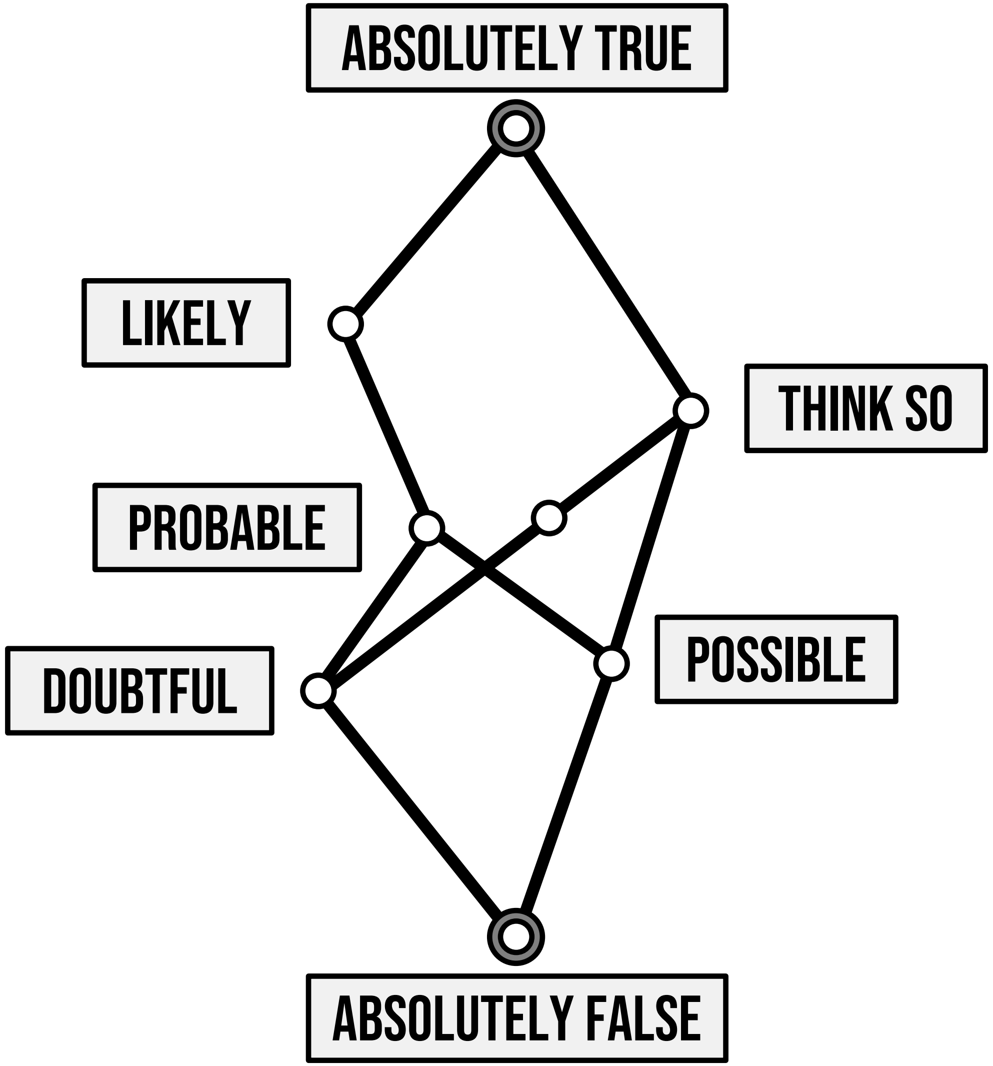
Individuals maintain personal structures for opinions, preferences, & perceptions



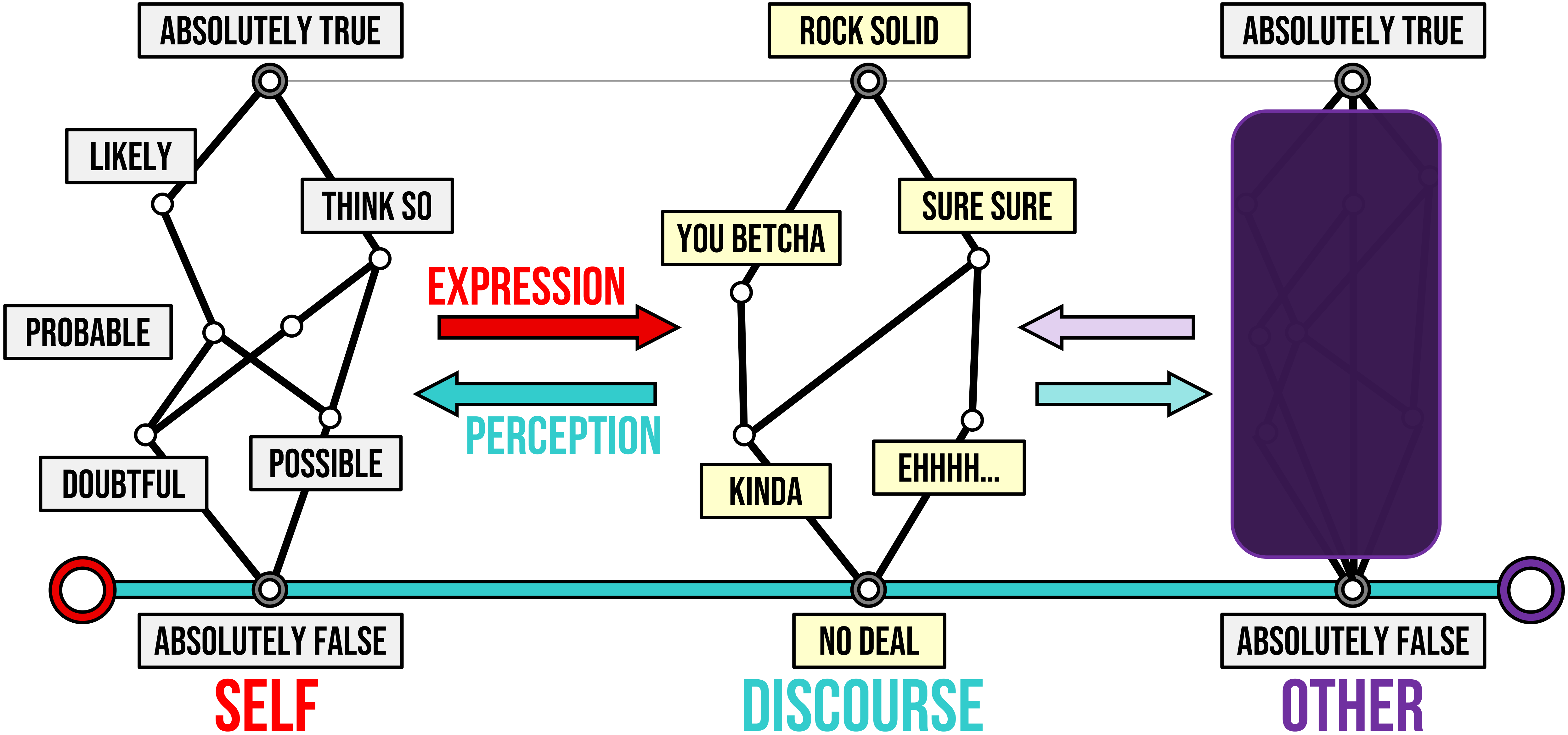




TRUTH & PERCEPTIONS & LATTICES



THE PROBLEM OF COMMUNICATION



EXAMPLE: COALITION-BUILDING

Consider a cover of a social network by neighborhoods N_v

Consider a sheaf $\mathcal{F}$ whose stalks correspond to the NERVE POWERSETS

$$\mathcal{F}(v) = 2^{N_v} \qquad \mathcal{F}(e) = 2^{N_u \cap N_v} \qquad \partial e = \{u, v\}$$

Vertex stalks $\mathcal{F}(v)$ are private estimates of who supports the coalition

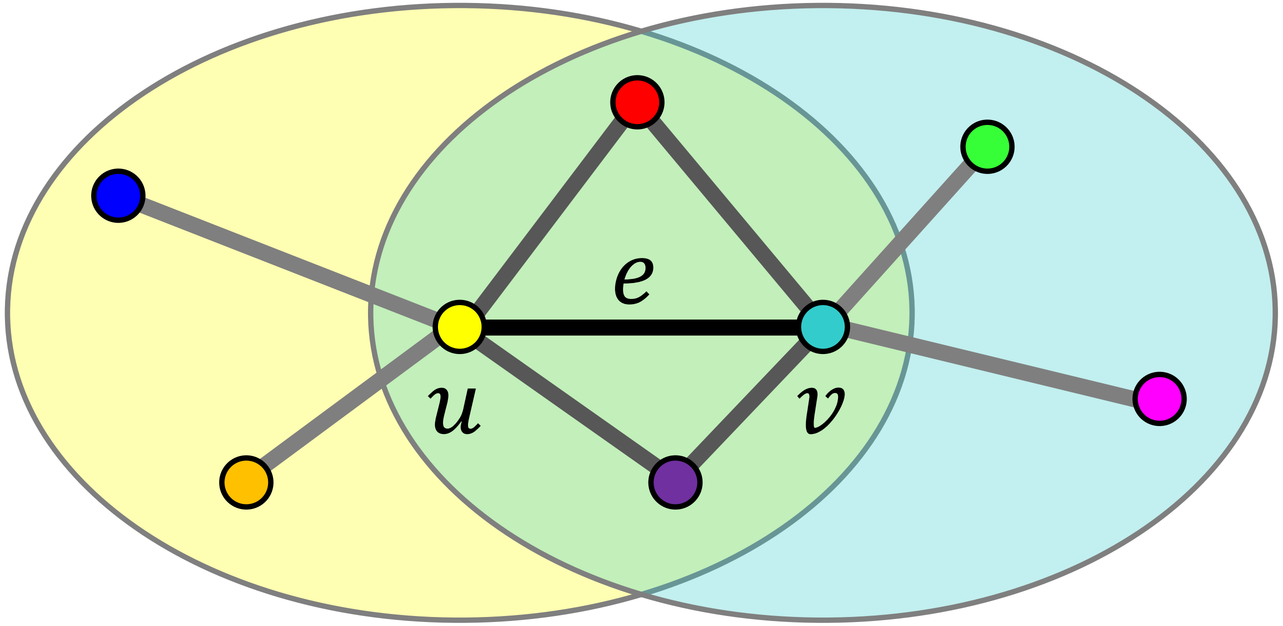
Maps to edge stalks $\mathcal{F}(e)$ are pairwise expressions of membership estimates

Think about how one could grow a coalition

EXAMPLE: COALITION-BUILDING

$\mathcal{F}(v) = 2^{N_v}$

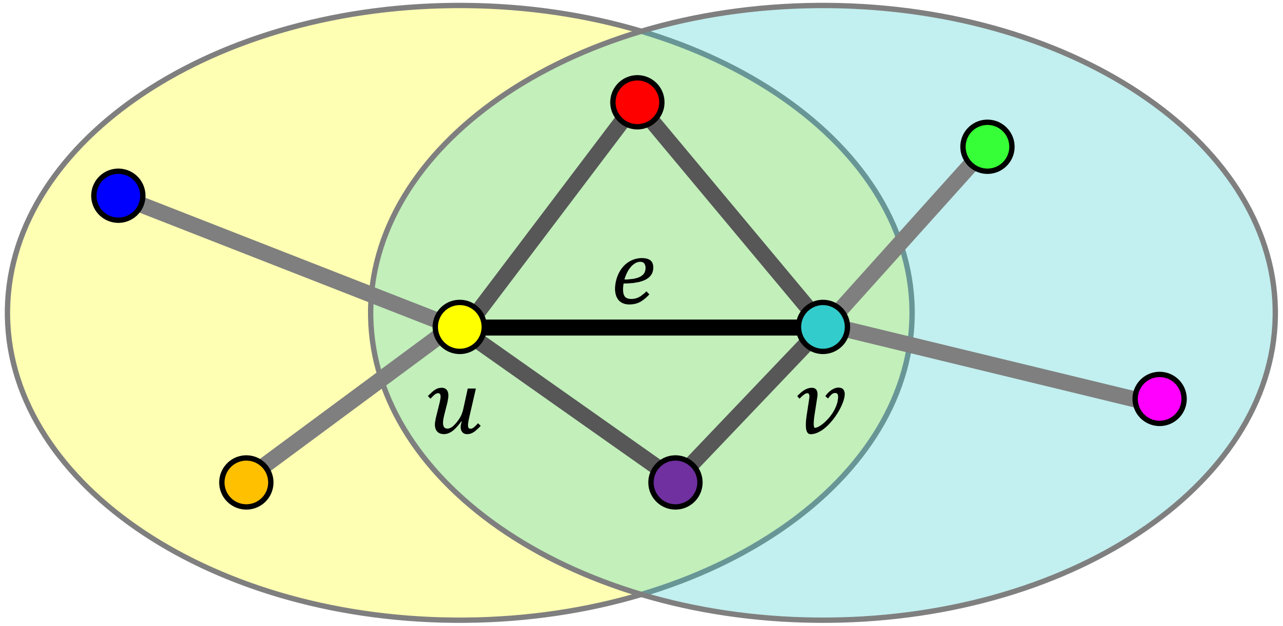
$\mathcal{F}(e) = 2^{N_u \cap N_v}$



Think about how one could grow a coalition

EXAMPLE: COALITION-BUILDING

Expression/Inference can be more subtle than *"Are you for or against?"*



Think about how one could grow a coalition

COMMUNICATION IN LATTICES

IS A BIT SUBTLE...

LATTICE MORPHISMS

GALOIS CONNECTIONS

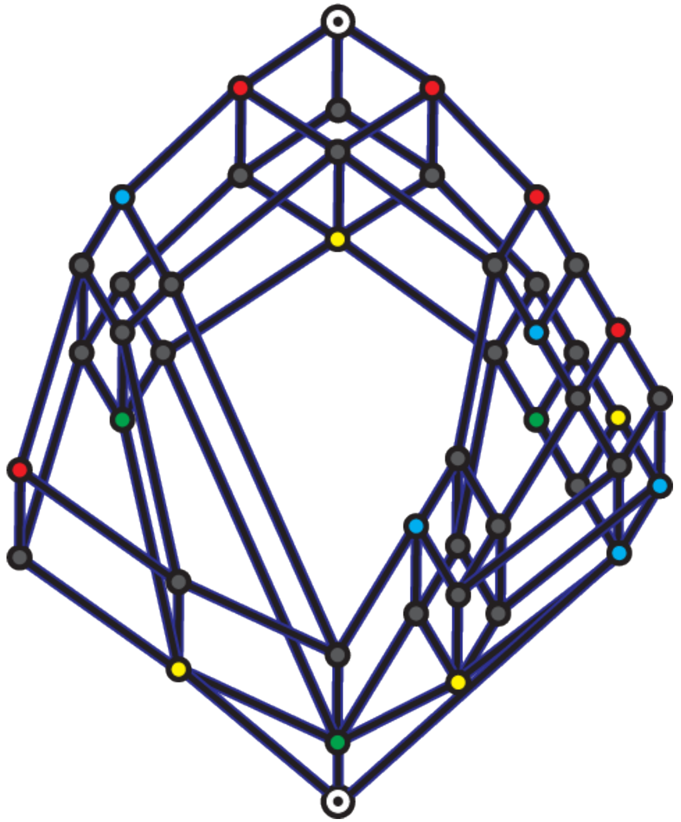
LATTICE CONNECTIONS come in ADJOINT PAIRS

$$X \underset{f^{\bullet}}{\overset{f_{\bullet}}{\rightleftarrows}} Y$$

$$f_{\bullet}(x) \leq y \iff x \leq f^{\bullet}(y)$$

$$f_{\bullet}(x) = \bigwedge f^{\bullet -1}(x^{\uparrow})$$

$$f^{\bullet}(y) = \bigvee f_{\bullet}^{-1}(y^{\downarrow})$$



WE WANT GLOBAL SECTIONS

BUT WE CAN'T JUST "RUN THE HEAT EQUATION" ON THESE SHEAVES
LIKE WE COULD DO FOR VECTOR-VALUED SYSTEMS

DEFINITION: TARSKI LAPLACIAN

DEFINITION : Let $\mathcal{F}$ be a NETWORK SHEAF of LATTICES
Let $\mathrm{x} = (x_v) \in C^0$ be a 0-COCHAIN

TARSKI LAPLACIAN

$$L : C^0\mathcal{F} \to C^0\mathcal{F}$$

$$(L\mathbf{x})_v = \bigwedge_{e \in \delta v} (\mathcal{F}_{v \trianglelefteq e})^{\bullet} \left(\bigwedge_{w \in \partial e} (\mathcal{F}_{w \trianglelefteq e})_{\bullet} (x_w) \right)$$

WHY IS THIS A "LAPLACIAN"?

IT CERTAINLY DOESN'T LOOK LIKE IT…

DEFINITION: TARSKI LAPLACIAN

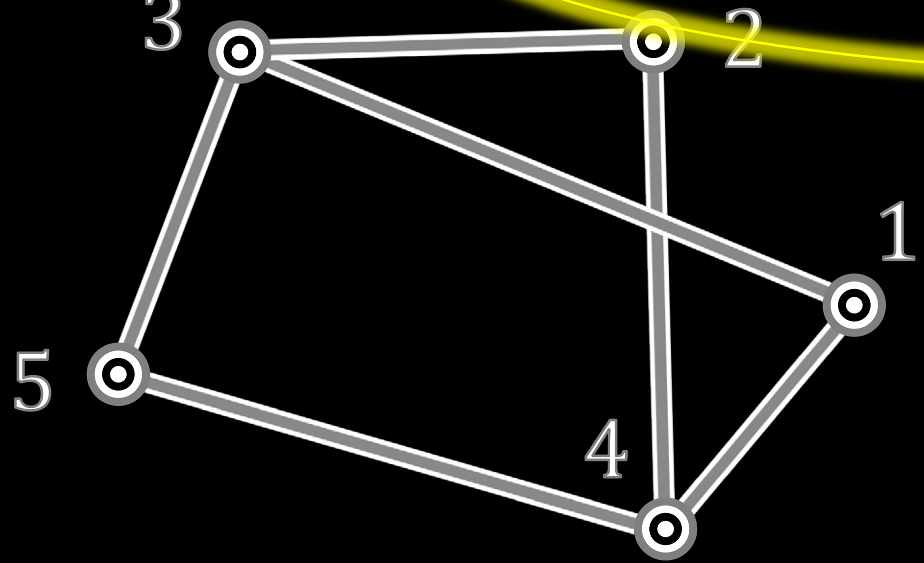
LEMMA : the TARSKI LAPLACIAN decomposes as

$$(L\mathbf{x})_v = \bigwedge_{e \in \delta v} (\mathcal{F}_{v \trianglelefteq e})^{\bullet} \left(\bigwedge_{w \in \partial e} (\mathcal{F}_{w \trianglelefteq e})_{\bullet} (x_w) \right)$$

$$= \underbrace{\left(\bigwedge_{e \in \delta v} (\mathcal{F}_{v \trianglelefteq e})^{\bullet} (\mathcal{F}_{v \trianglelefteq e})_{\bullet} (x_v) \right)}_{\text{EXPANSION}} \wedge \underbrace{\left(\bigwedge_{\substack{e \in \delta v \\ w \in \partial e - \{v\}}} (\mathcal{F}_{v \trianglelefteq e})^{\bullet} (\mathcal{F}_{w \trianglelefteq e})_{\bullet} (x_w) \right)}_{\text{MIXING}}$$

the Graph Laplacian

$$L = D - A = BB^{\top}$$



$$L = \begin{bmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Very Useful Indeed!

DEFINITION: TARSKI LAPLACIAN

LEMMA : the TARSKI LAPLACIAN decomposes as

$$(L\mathbf{x})_v = \bigwedge_{e \in \delta v} (\mathcal{F}_{v \trianglelefteq e})^{\bullet} \left(\bigwedge_{w \in \partial e} (\mathcal{F}_{w \trianglelefteq e})_{\bullet} (x_w) \right) \qquad \mathbf{BB^T}$$

$$= \underbrace{\left(\bigwedge_{e \in \delta v} (\mathcal{F}_{v \trianglelefteq e})^{\bullet} (\mathcal{F}_{v \trianglelefteq e})_{\bullet} (x_v) \right)}_{\text{EXPANSION}} \wedge \underbrace{\left(\bigwedge_{\substack{e \in \delta v \\ w \in \partial e - \{v\}}} (\mathcal{F}_{v \trianglelefteq e})^{\bullet} (\mathcal{F}_{w \trianglelefteq e})_{\bullet} (x_w) \right)}_{\text{MIXING}}$$

$$\mathbf{D - A}$$

DIFFUSION

TARSKI FIXED POINT THEOREM

THEOREM : For a COMPLETE LATTICE, X,
and an order-preserving endomorphism $f: X \to X$
the fixed point set $Fix(f)$ is a
NONEMPTY
COMPLETE
QUASI-SUBLATTICE
of X

HODGE-TARSKI FIXED POINT THEOREM

MAIN THEOREM : For a network sheaf $\mathcal{F}$ of COMPLETE LATTICES, the GLOBAL SECTIONS are computed via the TARSKI LAPLACIAN

$$H^0\mathcal{F} = Fix(Id \wedge L)$$

COMPARE : $x_{n+1} = (I - L)x_n$

COMPARE : $dx/dt = -Lx$

COROLLARY : $H^0\mathcal{F}$ is a NONEMPTY COMPLETE QUASI-SUBLATTICE of $C^0\mathcal{F}$

HIGHER DIMENSIONS

HODGE-TARSKI FIXED POINT THEOREM

THEOREM : For a network sheaf $\mathcal{F}$ of COMPLETE LATTICES, the GLOBAL SECTIONS are computed via the TARSKI LAPLACIAN

$$H^0\mathcal{F} = Fix(Id \wedge L)$$

THINK : the global sections are really the 0-dimensional sheaf cohomology

DOES HIGHER DIMENSIONAL SHEAF COHOMOLOGY MAKE SENSE?

RECALL: TARSKI LAPLACIAN

TARSKI LAPLACIAN

$$L : C^0\mathcal{F} \to C^0\mathcal{F}$$

$$(L\mathbf{x})_v = \bigwedge_{e \in \delta v} (\mathcal{F}_{v \trianglelefteq e})^{\bullet} \left(\bigwedge_{w \in \partial e} (\mathcal{F}_{w \trianglelefteq e})_{\bullet} (x_w) \right)$$

EASY! Replace vertices with k-cells and edges with cofaces.

$$L_k : C^k\mathcal{F} \to C^k\mathcal{F}$$

HODGE-TARSKI COHOMOLOGY

DEFINITION : For a cellular sheaf $\mathcal{F}$ of complete lattices on a cell complex, the TARSKI COHOMOLOGY is defined via the TARSKI LAPLACIAN

$$TH^k\mathcal{F} := Fix(Id \wedge L_k)$$

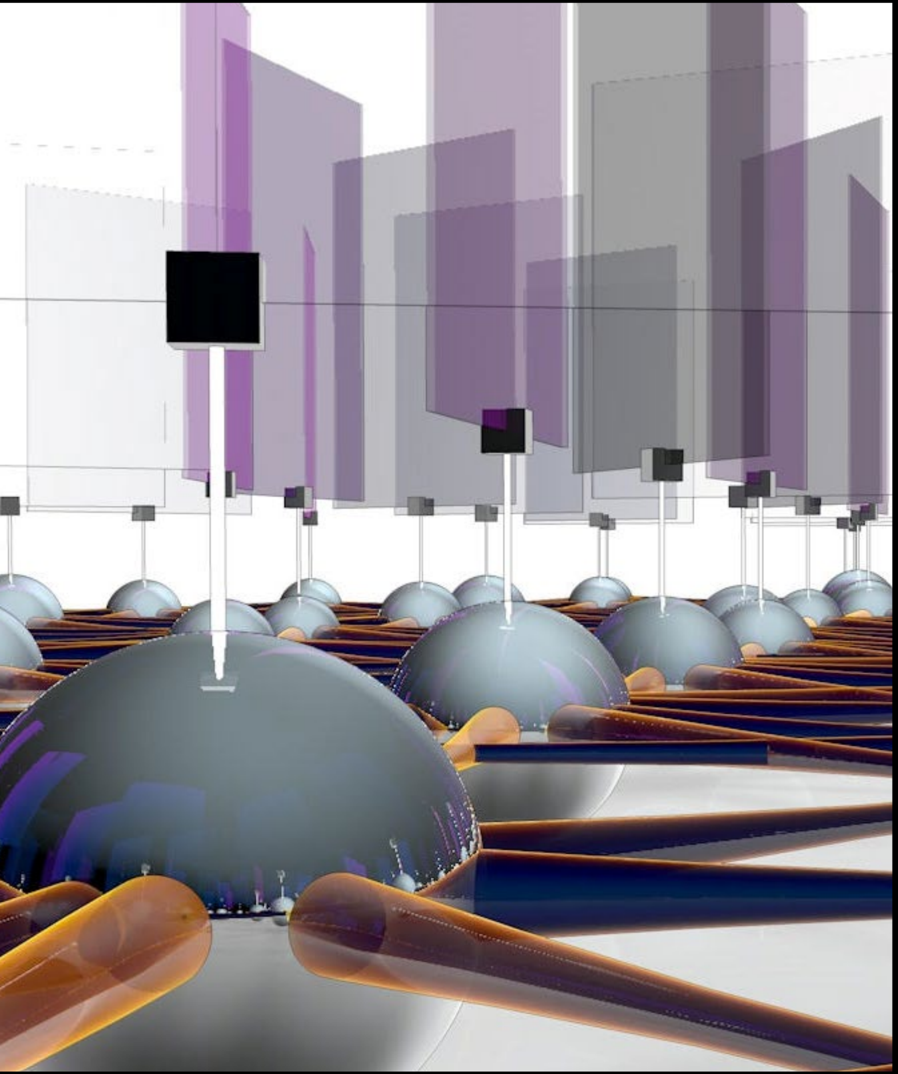
THEOREM : $TH^k\mathcal{F}$ IS A NONEMPTY COMPLETE QUASI-SUBLATTICE OF $C^k\mathcal{F}$

WHAT IS THIS?

THAT'S A GOOD QUESTION…

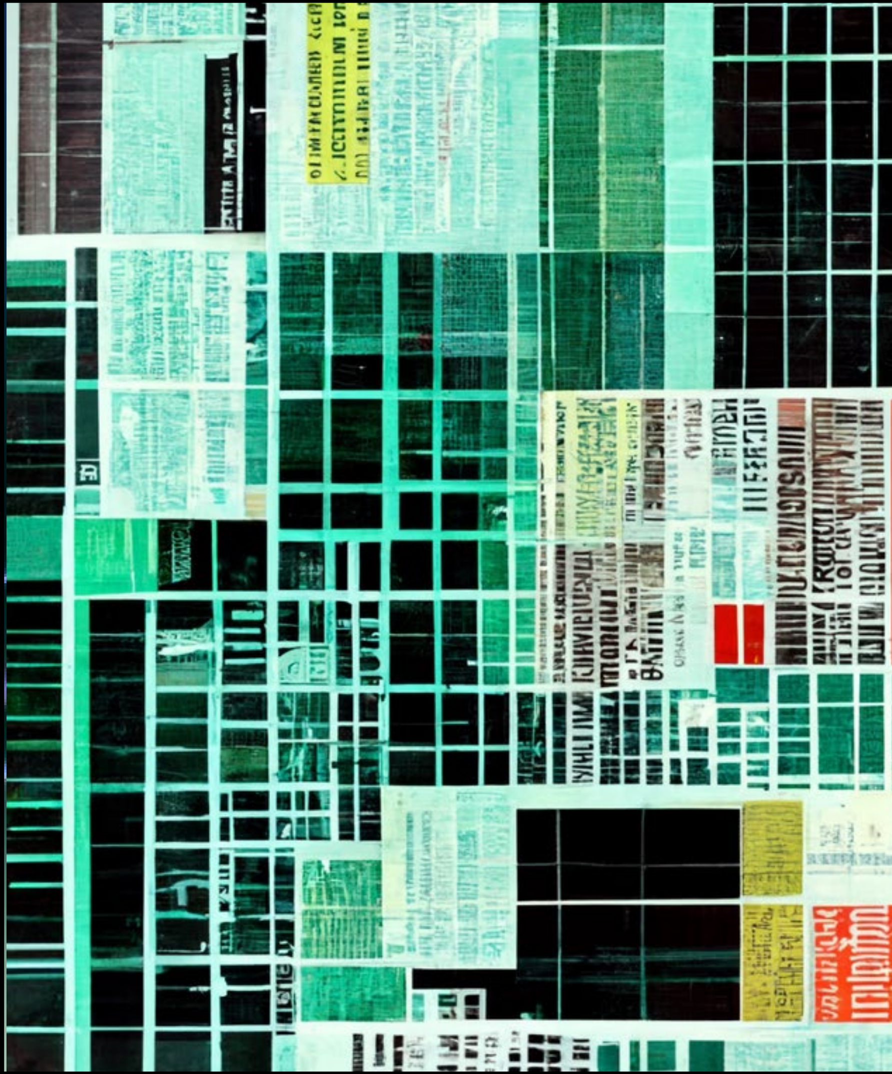
LET'S GO TO THE FUTURE ...

THREE PRINCIPAL TOOLS



SHEAVES

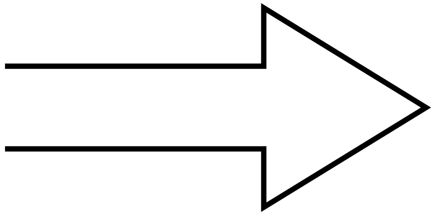
LATTICES

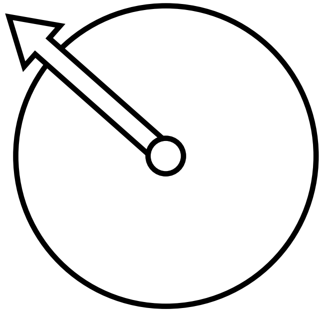


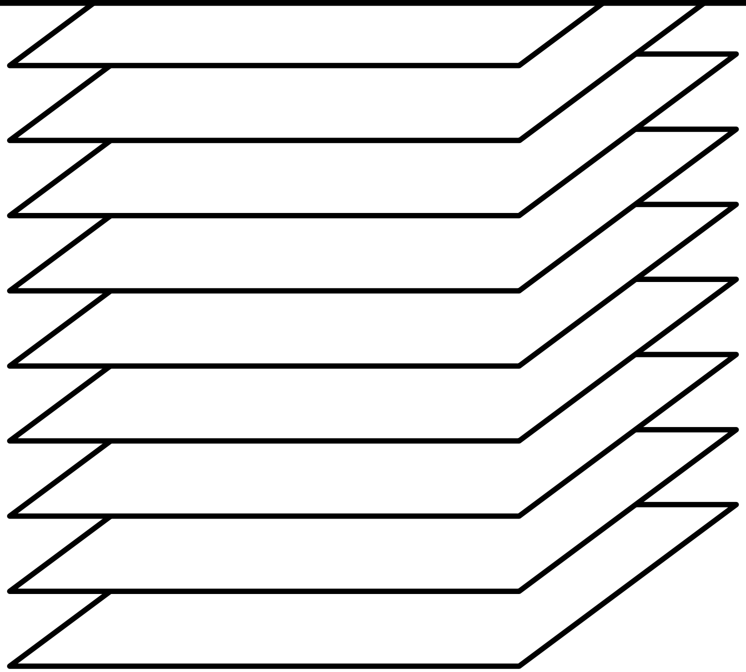
LAPLACIANS

TOWARDS REASONING

ADDITIONAL / HIGHER LATTICE STRUCTURES







SEMANTIC

RESIDUATED

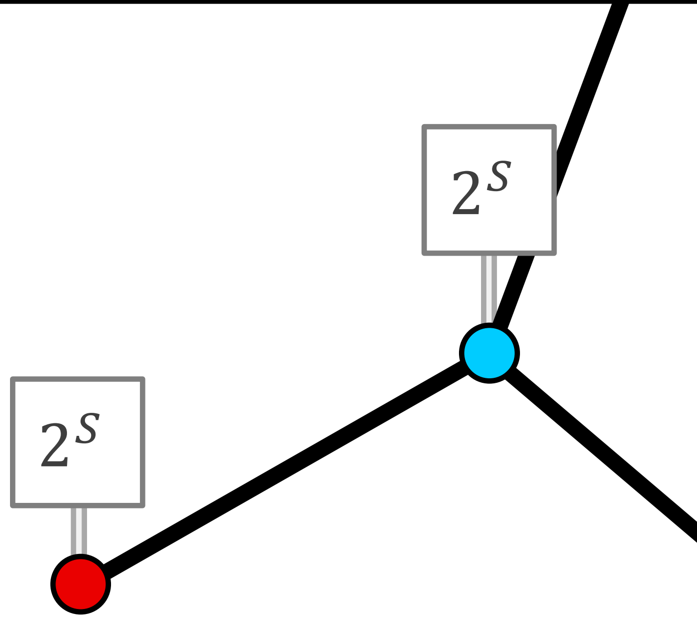
CATEGORICAL

KRIPKE SEMANTICS & DISTRIBUTED REASONING

GIVEN : Kripke frame = $(S, \mathcal{K}_i \subset S \times S)$

DERIVE : Galois connections on powerset of states S

$$2^S \underset{K_i^{\forall}}{\overset{K_i^{\exists}}{\rightleftarrows}} 2^S$$



SEMANTIC SHEAF : A graph of agents contemplating propositions S, communicating over the network to neighbors about knowledge/beliefs…

KRIPKE SEMANTICS & DISTRIBUTED REASONING

SEMANTIC SHEAF : A graph of agents contemplating propositions S, communicating over the network to neighbors about knowledge/beliefs…

DUAL TARSKI LAPLACIANS : For knowledge/possibility consensus

$$(L\sigma)_i = \bigwedge_{j\to i} \mathcal{K}_i^{\forall}\mathcal{K}_j^{\exists}(\sigma_j)$$

$$(L^*\sigma)_i = \bigvee_{j\to i} \mathcal{K}_j^{\exists}\mathcal{K}_i^{\forall}(\sigma_j)$$

HODGE-TARSKI THEOREM : Iteration yields knowledge (resp. possibility) consensus over the network…

WORK IN PROGRESS

W/ PAIGE RANDALL NORTH : HANS RIESS : MIGUEL LOPEZ

THE PLAN...

LATTICE : of propositions and implications via up-sets

FUZZY LATTICE THEORY : strength of implication

cf. work of R. Bêlohlávek (1999)

NEW INGREDIENT = ENRICHED CATEGORY THEORY

BOOLEAN LATTICE = category enriched in $\{0,1\}$ with all finite meets/joins

FUZZY LATTICE = category enriched in a commutative ordered monoid with all finite meets/joins (e.g., $[0,1]$ for the classic residuated theory)

WEIGHTED LIMITS/COLIMITS -->> Laplacian & Hodge-Tarski fixed point theorem

OPEN QUESTIONS/DIRECTIONS

BEYOND NETWORKS : higher-dimensional complexes & cohomologies

BEYOND DIFFUSION : waves, reaction-diffusion, patterns, drift, …

BEYOND LATTICES : coefficients are the programming language of a sheaf

BEYOND SHEAVES : stacks! & more…

BEYOND MATHEMATICS : so much for us to do in AI/ML/GSP +

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