CELLULAR SHEAVES, COHOMOLOGY, & APPLICATIONS

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PART 2

OUTLINE

REASONING WITH SHEAVES

NONABELIAN GOALS

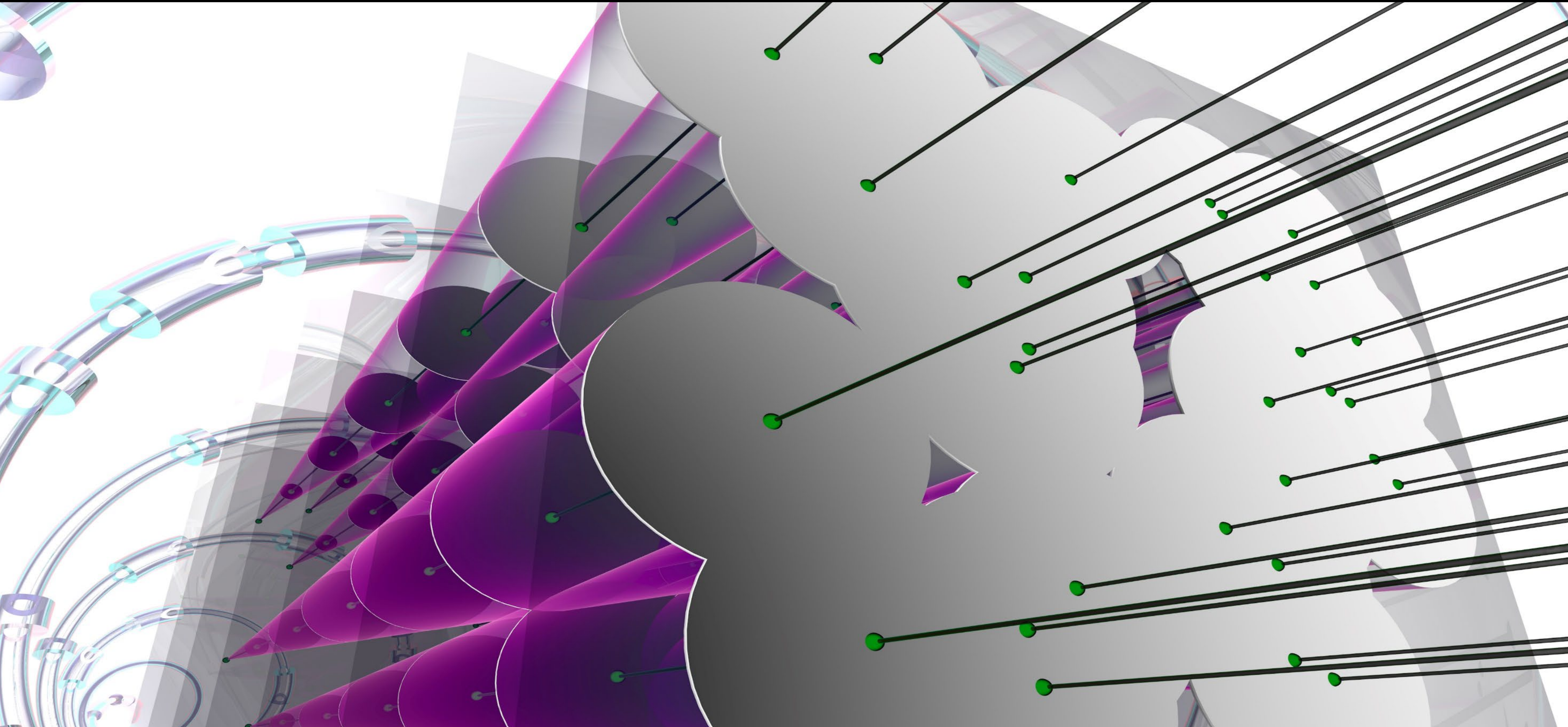
HARMONIC CONVERGENCE

SIMPLE EXAMPLES

CELLULAR SHEAVES

PROLOGUE : CALCULUS

WHAT'S USEFUL ABOUT SHEAVES?



SHEAF THEORY

GLOBAL SECTION FUNCTOR, $H^0(-)$

COLLATES ALL SOLUTIONS TO THE CONSTRAINTS IMPOSED BY THE SHEAF STALKS AND RESTRICTION MAPS;
GIVES AN ALGEBRAIC FORM TO THE SOLUTION SET.

COHOMOLOGY, $H^\bullet(-)$

$H^\bullet(-)$ CHARACTERIZES CONSTRAINT SATISFACTION AS A FUNCTION OF BOTH THE DOMAIN TOPOLOGY
AND THE ALGEBRA OF THE CONSTRAINTS.

MORPHISMS/PUSHFORWARDS/PULLBACKS

THESE OPERATIONS TRANSFORM SHEAVES (FROM ONE BASE SPACE TO ANOTHER)

HOM AND TENSOR PRODUCTS

HOM CLASSIFIES RELATIONSHIPS BETWEEN SHEAVES; $\otimes$ CONVOLVES SHEAF DATA.

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HOM AND TENSOR PRODUCTS

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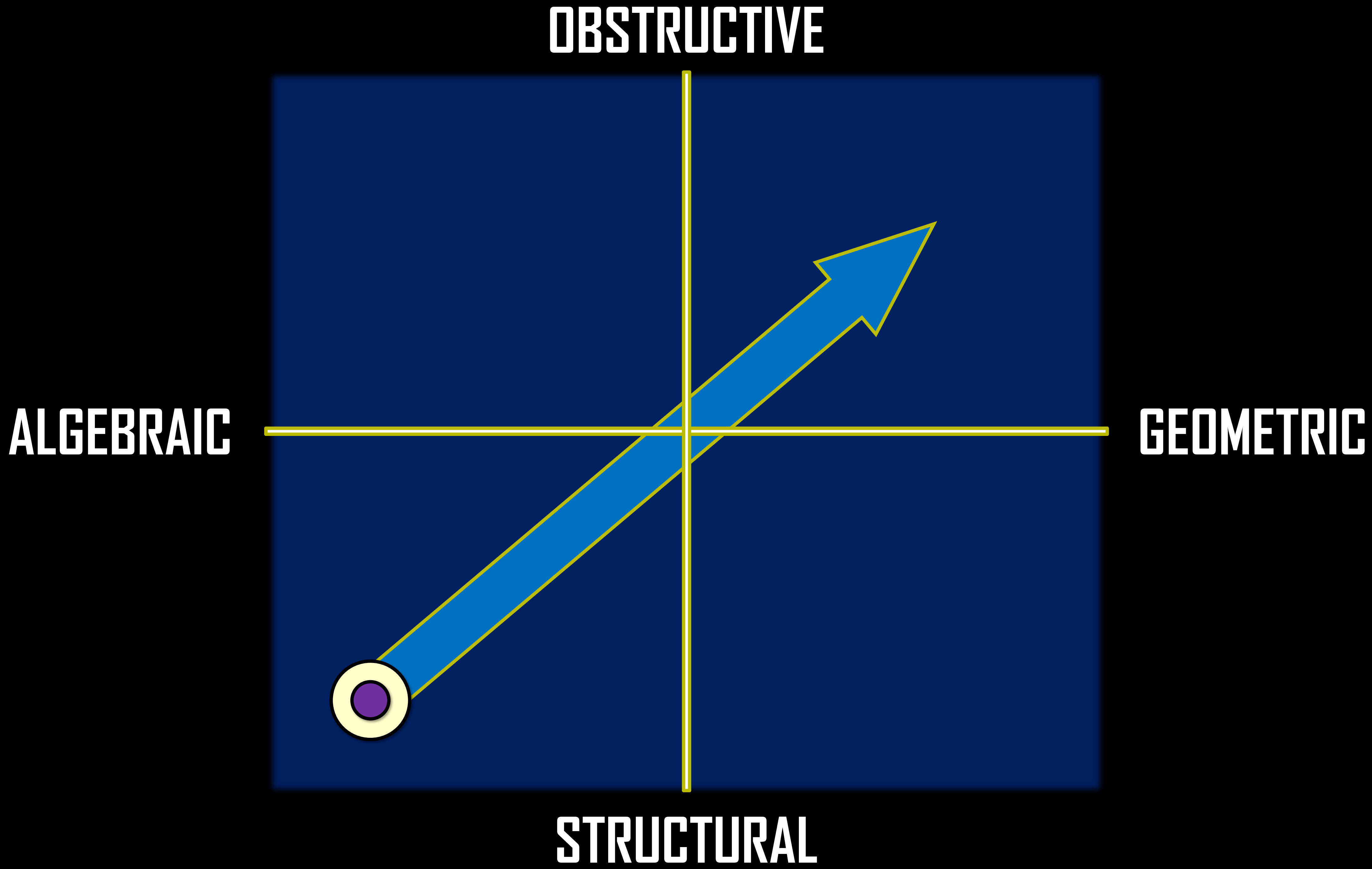
PROJECTIVE/INJECTIVE RESOLUTIONS

DECOMPOSES SHEAVES INTO SEQUENCES WITH NICE PROPERTIES

DERIVED FUNCTORS

COMBINATION OF COHOMOLOGY & SHEAF OPERATIONS <——— (POWER)

EXAMPLE: EULER INTEGRAL = A CERTAIN RIGHT DERIVED PUSHFORWARD


OBSTRUCTIVE
ALGEBRAIC
GEOMETRIC
STRUCTURAL


IT'S TIME FOR SOME GEOMETRY

BACKGROUND : LAPLACIANS

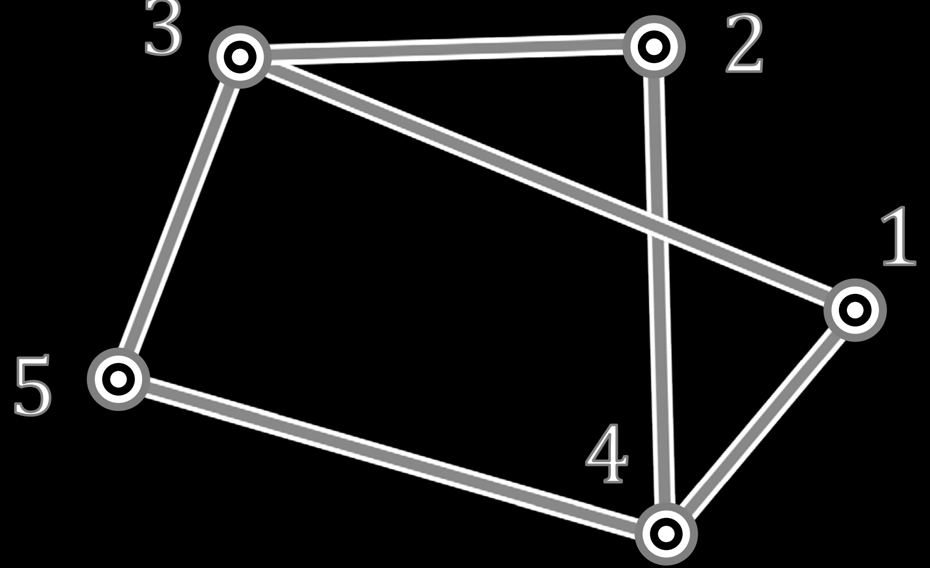
LAPLACIANS

COME IN MULTIPLE FLAVORS…

- ANALYTIC
- TOPOLOGICAL
- COMBINATORIAL

the Graph Laplacian

$$L = D - A = BB^{\top}$$



$$L = \begin{bmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Very Useful Indeed!

the Graph Laplacian

Spectral Graph Theory
Clustering & Consensus
Graph Signal Processing

Very Useful Indeed!

AGAIN WITH THE CALCULUS...

THE CLASSIC IDEA: HARMONIC DIFFERENTIAL FORMS

HODGE LAPLACIAN

$$\Delta = (d + d^*)^2 = d^*d + dd^*$$

$$\Omega^0(M) \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^1(M) \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^2(M) \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^3(M) \underset{d^*}{\overset{d}{\rightleftarrows}}$$

0-FORMS: Δ^0 acting on $\Omega^0(M)$; 1-FORMS: Δ^1 acting on $\Omega^1(M)$; 2-FORMS: Δ^2 acting on $\Omega^2(M)$; 3-FORMS: Δ^3 acting on $\Omega^3(M)$

FOR AN ORIENTABLE (COMPACT) FINITE-DIMENSIONAL MANIFOLD...

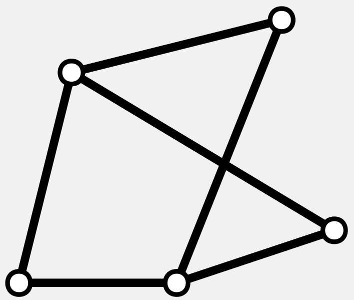
HODGE THEOREM: $\ker \Delta^k \cong H^k(M; \mathbb{R})$ – THE KERNEL of the LAPLACIAN COMPUTES COHOMOLOGY

YOU CAN UNDERSTAND THE TOPOLOGY OF A SMOOTH MANIFOLD VIA ITS HARMONIC FORMS

THE COMBINATORIAL PERSPECTIVE

RECALL: SPECTRAL GRAPH THEORY

EIGENVALUES of the GRAPH LAPLACIAN are IMPORTANT



$$L_G = \begin{pmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

GRAPH LAPLACIAN

$$(L_G x)_i = \sum_{i \sim j} w_{ij}(x_i - x_j)$$

SPECTRUM

$$\sigma_G = (0, 2, 2, 3, 5)$$

CLASSIC APPLICATION: # OF ZERO-EIGENVALUES = # OF GRAPH COMPONENTS

PARTITIONING; GRAPH SPARSIFICATION; DIMENSIONALITY REDUCTION; RANDOM WALKS; DISTRIBUTED OPTIMIZATION...

RECENT DEVELOPMENT of GRAPH CONNECTION LAPLACIAN

Bandeira-Singer-Spielman : Cheeger inequality yields spectral algorithm for synchronization with deterministic optimality bounds

Ye-Lim = cohomological perspective : Gao-Brozski-Mukherjee = vector bundle perspective

GRAPH CONNECTION LAPLACIAN

$$(L_G x)_i = \sum_{i \sim j} w_{ij}(x_i - \rho_{ij} x_j)$$

$$\rho_{ij} \in O_n$$

THESE all are SPECIAL CASES of SOMETHING SHEAFY...

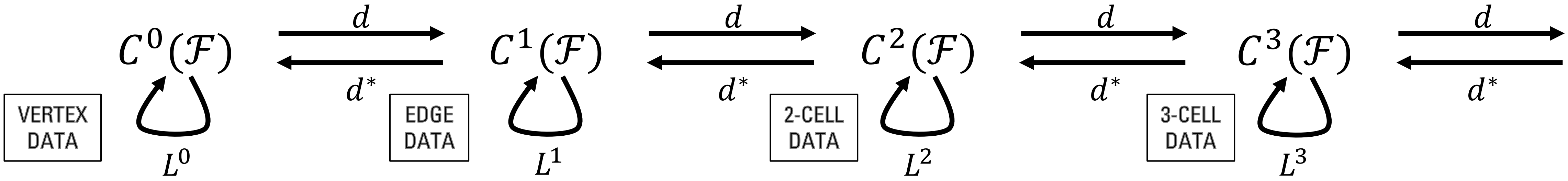
JOINT WORK WITH **JAKOB HANSEN**

SHEAF LAPLACICANS

THE CORE IDEA: EXTEND LAPLACIAN to CELLULAR SHEAVES

HODGE LAPLACIAN

$$L = (d + d^*)^2 = d^*d + dd^*$$



THIS IS TRULY a GENERALIZATION of the GRAPH LAPLACIAN

FOR X a GRAPH and $\mathcal{F}$ the CONSTANT SHEAF, L^0 is the GRAPH LAPLACIAN

IDEA : lift spectral graph theory to sheaves of vector spaces

J. Hansen + G

SPECTRAL SHEAF THEORY

BASIC RESULTS

THEOREMS: [JH-RG 2018]

1: $\ker L^k \cong H^k(X;\mathcal{F})$

2: $\langle u | L^0 u \rangle = \langle du | du \rangle$

$=$ DISTANCE TO u BEING GLOBAL SECTION

SPECTRAL SHEAF THEORY

FOUNDATIONS OF SPECTRAL SHEAF THEORY

CONDENSED LIST OF RESULTS

J. Hansen & R. Ghrist, "Towards a Spectral Theory of Sheaves"
J. Appl. Comput. Topology, 3(4), 315-358, 2019.

HARMONIC EXTENSION: for $A \subset X$ and $H^k(X, A; \mathcal{F}) = 0$, there is a **UNIQUE HARMONIC EXTENSION** of cochains on A to X

MAXIMUM MODULUS PRINCIPLE: for $\boldsymbol{O(n)}$ **BUNDLES** over a graph

EFFECTIVE RESISTANCE: a pair of **HOMOLOGOUS CYCLES** has effective resistance defined via minimal-norm BOUNDING CHAIN

SPARSIFICATION: using **EFFECTIVE RESISTANCE** as a probability on GRAPH CYCLES allows for RANDOM SAMPLING to COMPRESS a sheaf with CONTROL of SPECTRUM of SHEAF LAPLACIAN

EXPANDERS: an **η-EXPANDER SHEAF** is a k-REGULAR SHEAF whose ADJACENCY MATRIX has η-BOUNDED SPECTRUM

EXPANDER MIXING LEMMA: computes **EXPECTED TRACE** of EDGES between SUBSETS of VERTEX SET of a REGULAR SHEAF

SHEAF APPROXIMATION: using **SPARSE APPROXIMATIONS** to SHEAVES permits COMPRESSION and REDUCED DATA TRANSFER

OPTIMIZATION

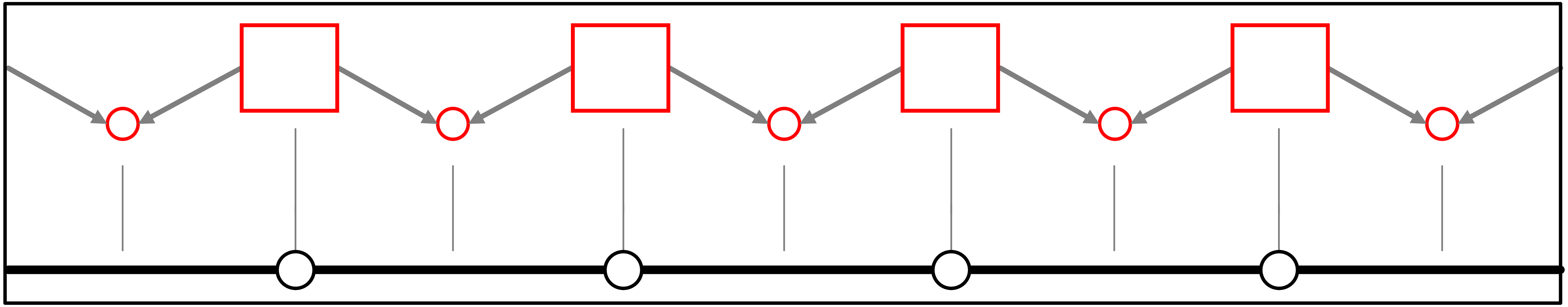
OPTIMIZATION

DISTRIBUTED OPTIMIZATION OVER SHEAVES WITH COHOMOLOGICAL CONSTRAINTS

Consider a graph G and a sheaf $\mathcal{H}$ of vector spaces over G

Let $F = \{f_v\}_{v \in V(G)}$ be a set of convex functionals on vertices of G

PROBLEM: minimize $F(x) = \sum_v f_v(x)$ subject to x being a global section of $\mathcal{H}$



COHOMOLOGY AS CONSTRAINT

OPTIMIZATION

DISTRIBUTED OPTIMIZATION OVER SHEAVES WITH COHOMOLOGICAL CONSTRAINTS

PROBLEM: minimize $F(x) = \sum_v f_v(x)$ subject to x being a global section of $\mathcal{H}$

SOLUTION: form a Lagrangian $\mathcal{L}(x,\lambda) := F(x) + x^T L_{\mathcal{H}} x + \lambda^T L_{\mathcal{H}} \lambda$

Use continuous-time primal-dual evolution using ***sheaf Laplacians***

$$\dot{x} = -\nabla F - 2L_{\mathcal{H}} x - 2L_{\mathcal{H}} \lambda$$
$$\dot{\lambda} = L_{\mathcal{H}} x$$

CF. OZDAGLAR + AL.
JADBABAIE + AL.

LEMMA: asymptotic convergence to primal-dual KKT solutions

COROLLARY: all computations ***local and distributable***

EXTENSIONS: complexes; other constraints, such as fixing the 1-cochain image of δ

COHOMOLOGY AS CONSTRAINT

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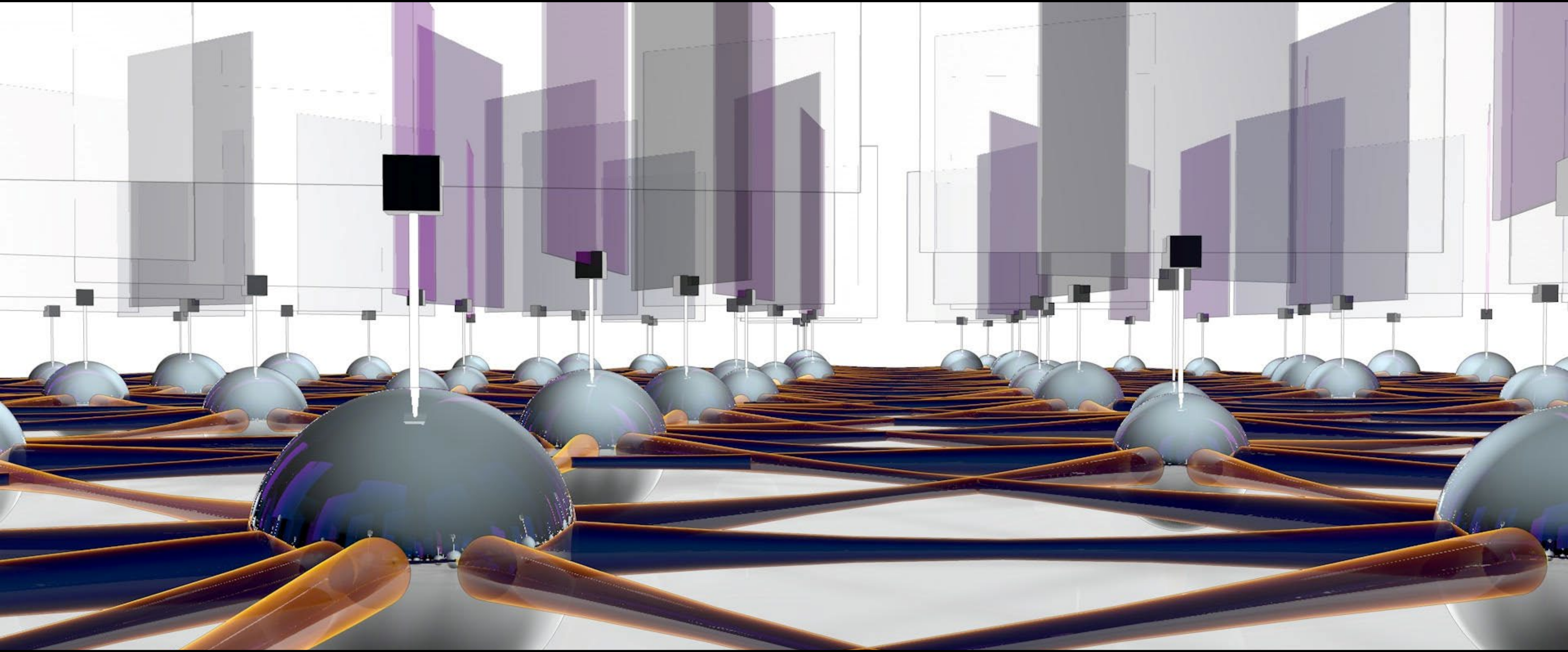
THIS IS JUST THE BEGINNING

OF WHAT MAY BE A VERY SIGNIFICANT & USEFUL SET OF TOOLS

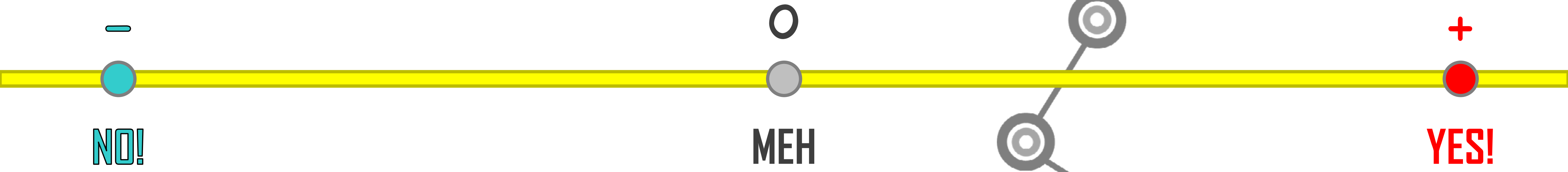
HOMOLOGICAL PROGRAMMING

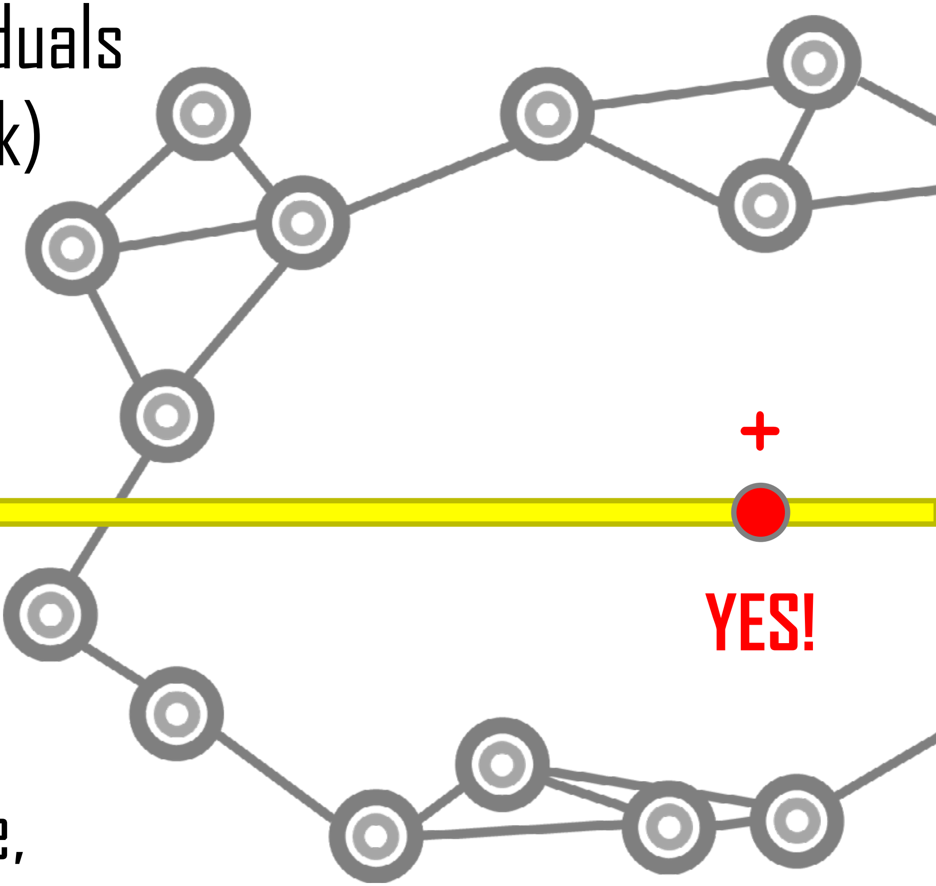


SHEAVES & DYNAMICS

OPINION DYNAMICS

CLASSIC : Let's say you have a social network of individuals
Model the network as an undirected graph (cf. FaceBook)
Assume that everybody has an opinion on some topic...





PROBLEM: What happens to peoples' opinions over time, assuming mutual influence?

THE CLASSICAL RESULT

[TAYLOR : 1968] Uses the GRAPH LAPLACIAN to predict change in opinions over time

Let x be the vector of opinions over the nodes of the social network.

$$\frac{d}{dt}x(t) = -\alpha L x(t)$$

$$\alpha > 0$$

$$x_{n+1} - x_n = -\alpha L\, x_n$$

THE CLASSICAL RESULT

[TAYLOR : 1968] Uses the GRAPH LAPLACIAN to predict change in opinions over time

Let x be the vector of opinions over the nodes of the social network.

$$\frac{d}{dt}x(t) = -\alpha L x(t) \qquad \alpha > 0$$

$$x_{n+1} = (I - \alpha L)x_n$$

THEOREM: Every initial condition evolves to ***consensus*** : locally-constant solutions

THE CLASSICAL RESULT

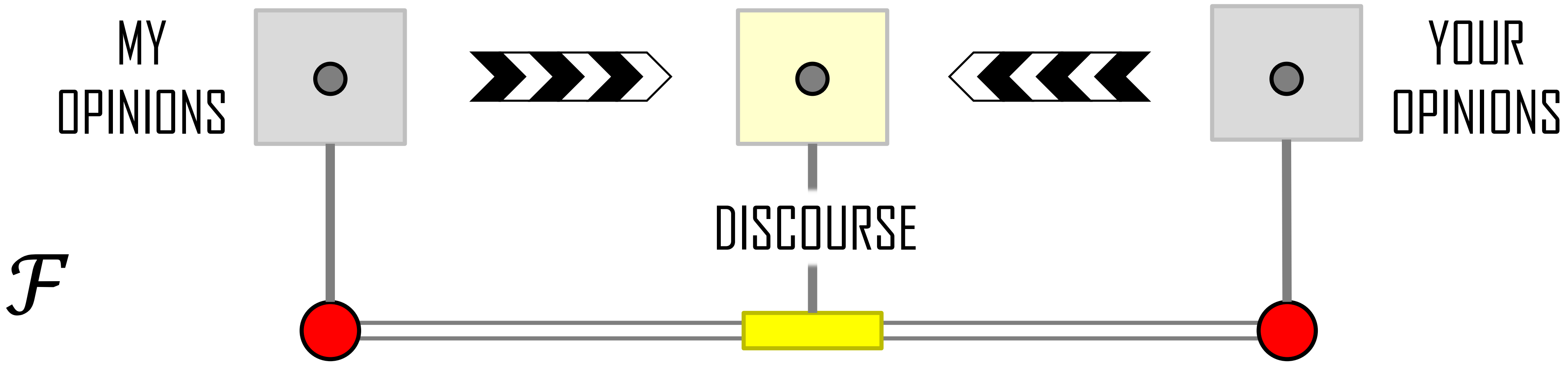
SO WE SEE WITH INCREASED CONNECTIVITY
A CORRESPONDING INCREASE IN CONSENSUS

THEOREM: Every initial condition evolves to ***consensus*** : locally-constant solutions

RECALL : DISCOURSE SHEAVES

J. Hansen + G

CONSIDER THE FOLLOWING MODEL ON A SOCIAL NETWORK



VERTEX STALKS : **OPINION SPACES** : private "basis" opinions from which policies are formulated

EDGE STALKS : **DISCOURSE SPACES** : public "basis" topics on which opinions are expressed

VERTEX-EDGE MAPS: **EXPRESSIONS** : how individuals choose to formulate opinions from bases

LET'S WORK WITH THE SHEAF LAPLACIAN

AND SEE WHAT WE CAN DO IN THIS MORE GENERAL SETTING

COHOMOLOGY & OPINION DISTRIBUTIONS

$C^0(\mathcal{F})$	0 - COCHAINS	opinion distributions (private)
$C^1(\mathcal{F})$	1 - COCHAINS	pairwise discussions (public)
$\delta: C^0 \rightarrow C^1$	COBOUNDARY	aggregate public disagreement
$L(C^0\mathcal{F})$	LAPLACIAN	average private disagreement
$H^0(\mathcal{F})$	GLOBAL SECTIONS	literally : harmonic opinions

HARMONIC OPINION DISTRIBUTIONS CLASSIFY EXPRESSED AGREEMENT

HARMONIC CONVERGENCE

PRINCIPAL THEOREM

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THEOREM : USING THE SHEAF LAPLACIAN FOR A HEAT EQUATION ON OPINIONS

$$\frac{d}{dt}x(t) = -\alpha L x(t) \qquad \alpha > 0$$

$$x_{n+1} = (I - \alpha L)x_n$$

EVERY INITIAL CONDITION $x_0 \in C^0(\mathcal{F})$ CONVERGES EXPONENTIALLY TO THE CLOSEST HARMONIC DISTRIBUTION $x_\infty \in H^0(\mathcal{F})$

(via orthogonal projection to the space of global sections)

WHAT DOES THIS MEAN?

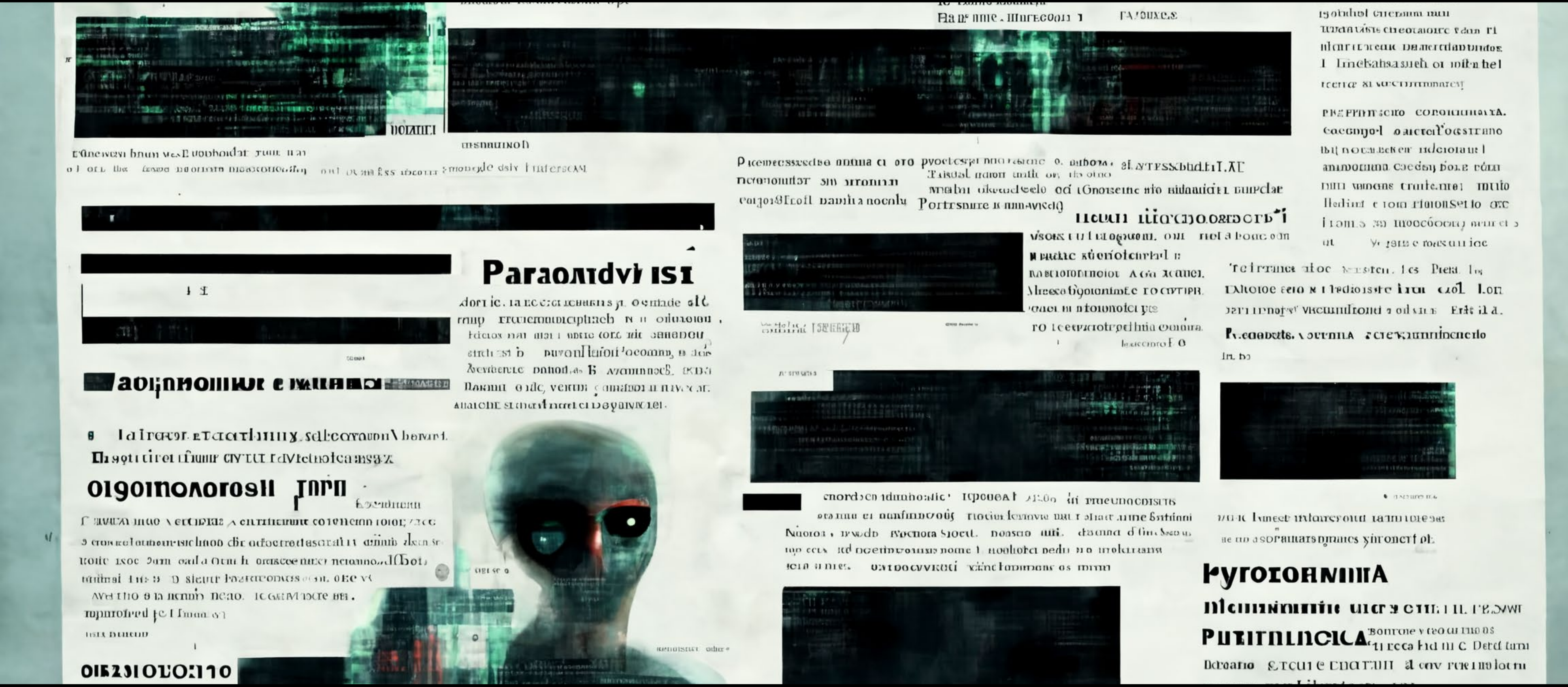
DOES EVERYONE AGREE?

HARMONIC EXTENSION

WHAT ABOUT STUBBORN AGENTS?

HOW DO WE ACCOUNT FOR PEOPLE WHO ARE NOT INFLUENCED BY OTHERS?

MOTIVATION : MIS-/DYS-INFORMATION



created via @midjourney AI with the prompt : paranoid information psyops

HARMONIC EXTENSION

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THEOREM : FOR THE MODIFIED HEAT EQUATION ON OPINION DISTRIBUTIONS

$$\frac{d}{dt}x(t) = \begin{cases} -\alpha L(x) & x \notin U \\ 0 & x \in U \end{cases} \qquad \alpha > 0$$

EVERY INITIAL CONDITION $x_0 \in C^0(\mathcal{F})$ CONVERGES TO THE CLOSEST **HARMONIC EXTENSION** OF $x|_U$

(such an extension always exists and is unique if $H^0(X, U; \mathcal{F}) = 0$)

PARAMETRIC STUBBORNESS

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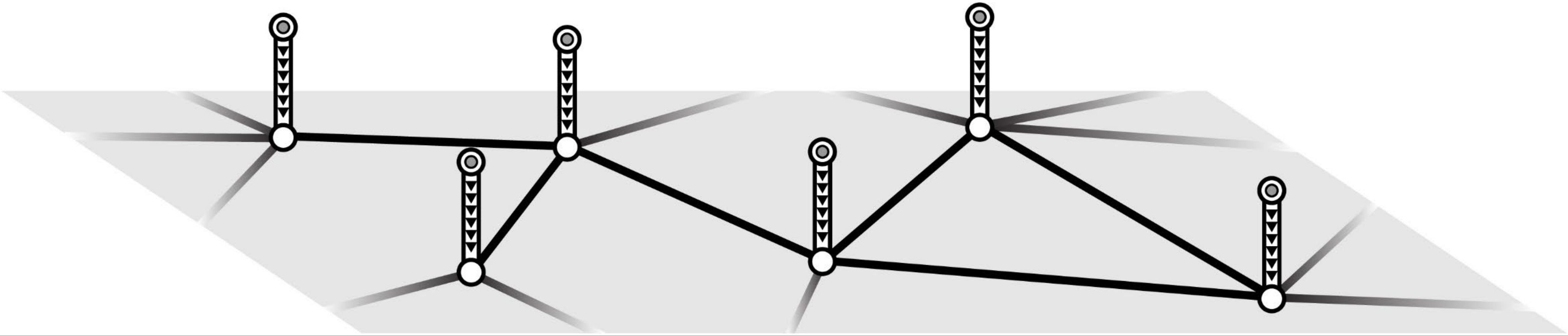
How to parametrize individuals' resistance to change?



Append a STUBBORN PRIOR to each agent & program customized stubbornness...

This yields tunable resistance-to-change for each agent.

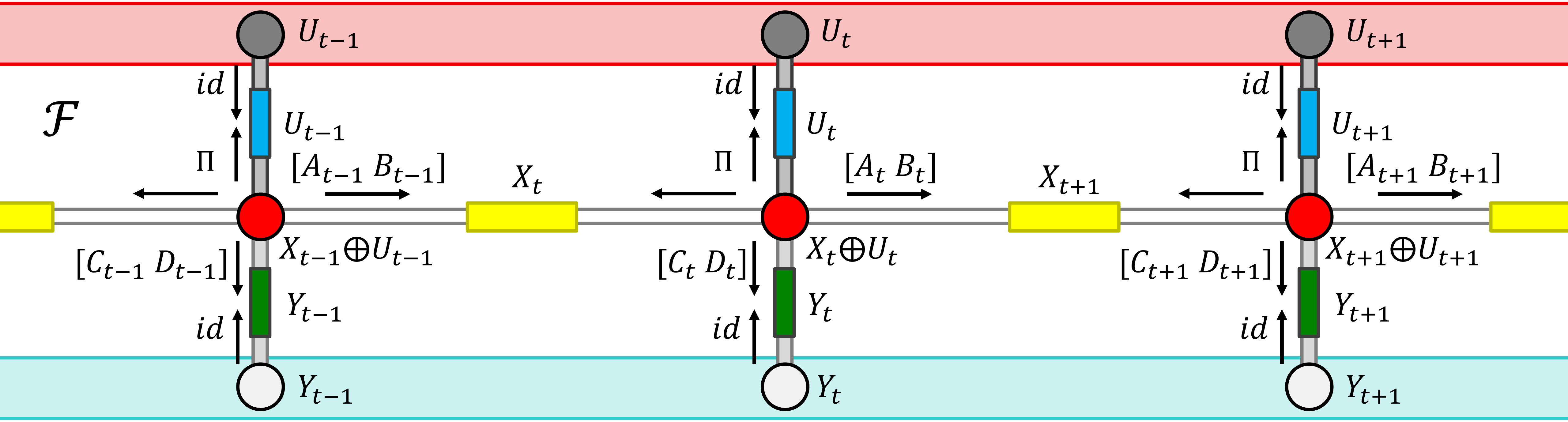
CONTROL THEORY

WHAT IF ONE PLANTS OPINIONS?

TO WHAT EXTENT CAN YOU DRIVE A PARTICULAR OUTCOME?

DISCRETE-TIME LINEAR SYSTEMS W/CONTROLS

Consider the modified system: $x_{t+1} = A_t x_t + B_t u_t$; $y_t = C_t x_t + D_t u_t$



Global sections: $H^0\mathcal{F}$ classifies global-time solutions

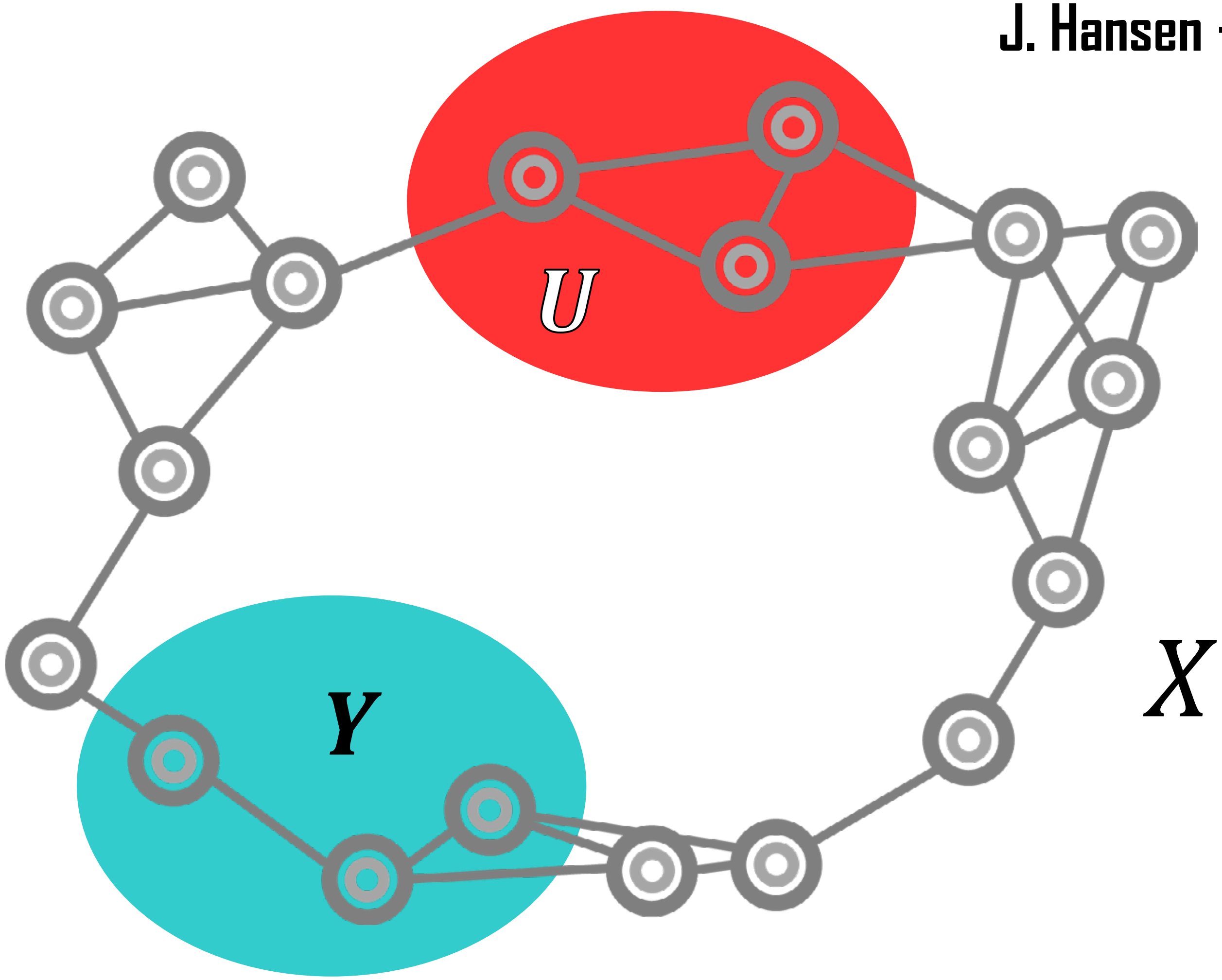
CONTROL THEORY

J. Hansen + G

IN A HARMONIC DISTRIBUTION…

$H^0(X, U; \mathcal{F})$ IS THE OBSTRUCTION TO CONTROLLABILITY

$H^0(X, Y; \mathcal{F})$ IS THE OBSTRUCTION TO OBSERVABILITY

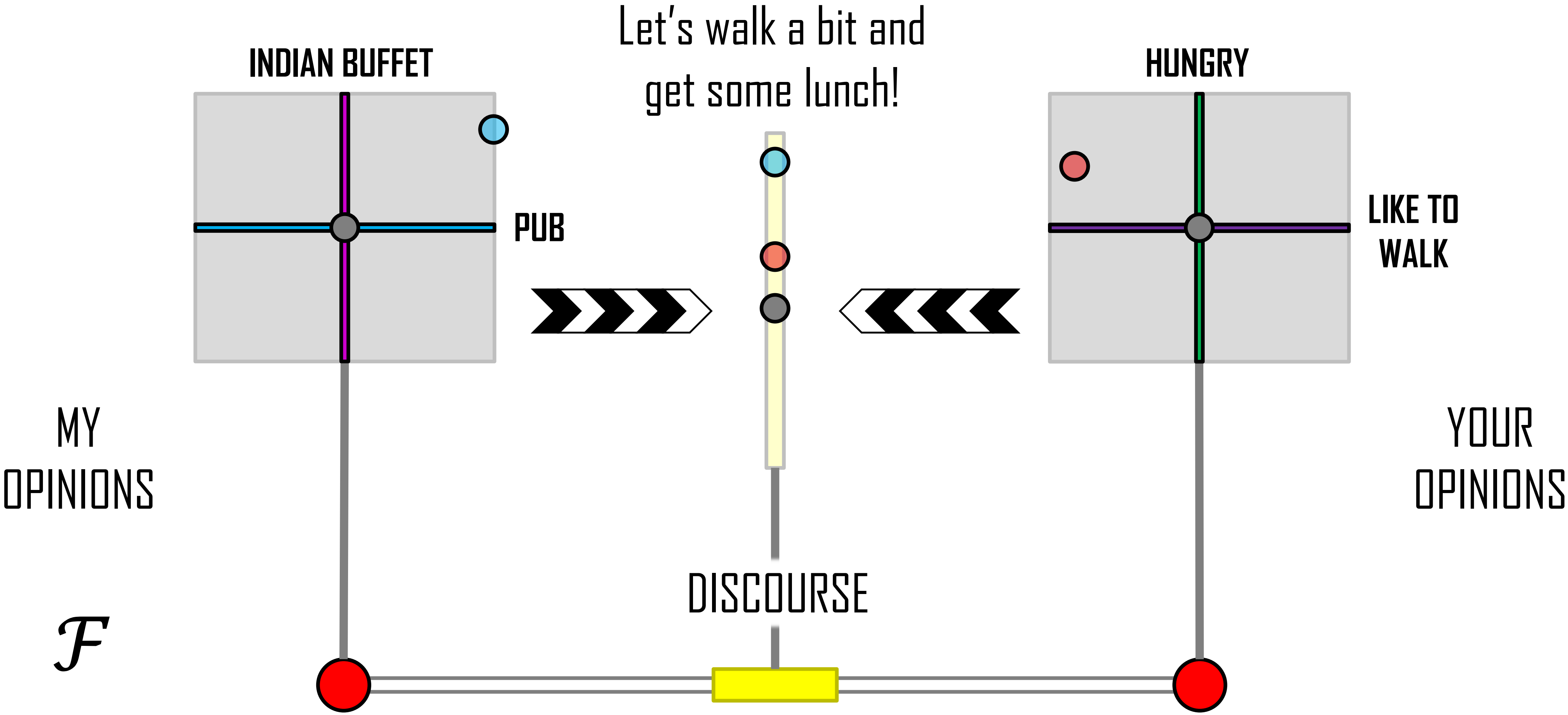


LEARNING HOW TO COMMUNICATE BETTER

HOW DO PEOPLE CHANGE?

IS IT REALISTIC TO POSIT THAT PEOPLES' OPINIONS CHANGE BASED ON DISCUSSION?

DISCOURSE SHEAVES EXAMPLE




EVOLVE IN THE SPACE OF SHEAVES

DIFFUSION ON DISCOURSE SHEAVES

J. Hansen + G

THEOREM : FOR EVERY INITIAL CONDITION $x_0 \in C^0(\mathcal{F})$

THE EXPRESSION-EVOLUTIONARY SYSTEM

$$\frac{d}{dt}\mathcal{F}_{v \trianglelefteq e}(t) = -\alpha(\mathcal{F}_{v \trianglelefteq e}x_v - \mathcal{F}_{u \trianglelefteq e}x_u)x_v^T \qquad \alpha > 0$$

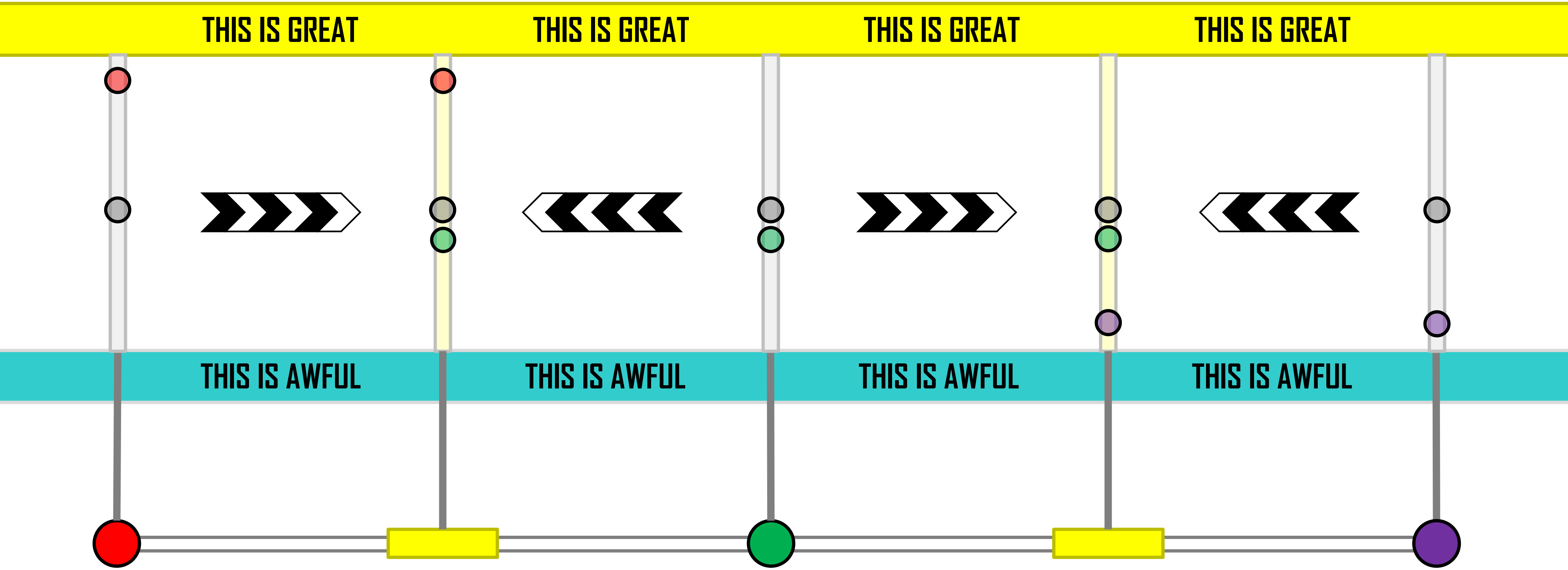
CONVERGES TO THE CLOSEST DISCOURSE SHEAF

FOR WHICH x_0 IS A GLOBAL SECTION

(where "closest" means in terms of squared Frobenius norm on the maps)

DISCOURSE SHEAVES

EXAMPLE



THIS IS A NICE BEGINNING...

VECTOR SPACES ARE A CONVENIENT TEST BED FOR THIS TYPE OF MODEL