CELLULAR SHEAVES, COHOMOLOGY, & APPLICATIONS

ROBERT GHRIST

ANDREA MITCHELL UNIVERSITY PROFESSOR
MATHEMATICS & ELECTRICAL/SYSTEMS ENGINEERING
UNIVERSITY OF PENNSYLVANIA

PART 1

OUTLINE

REASONING WITH SHEAVES

NONABELIAN GOALS

HARMONIC CONVERGENCE

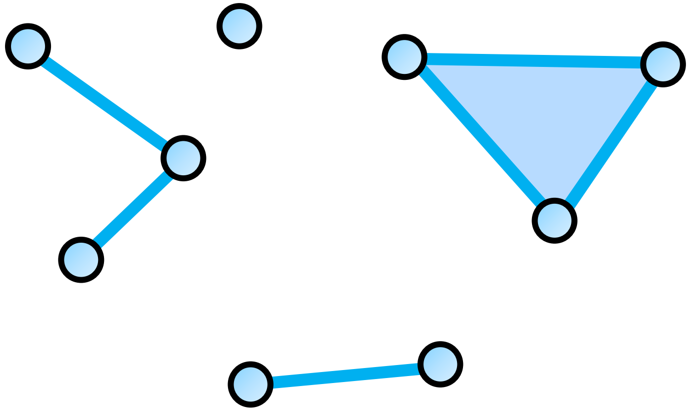
SIMPLE EXAMPLES

CELLULAR SHEAVES

PROLOGUE : CALCULUS

EULER COUNTING

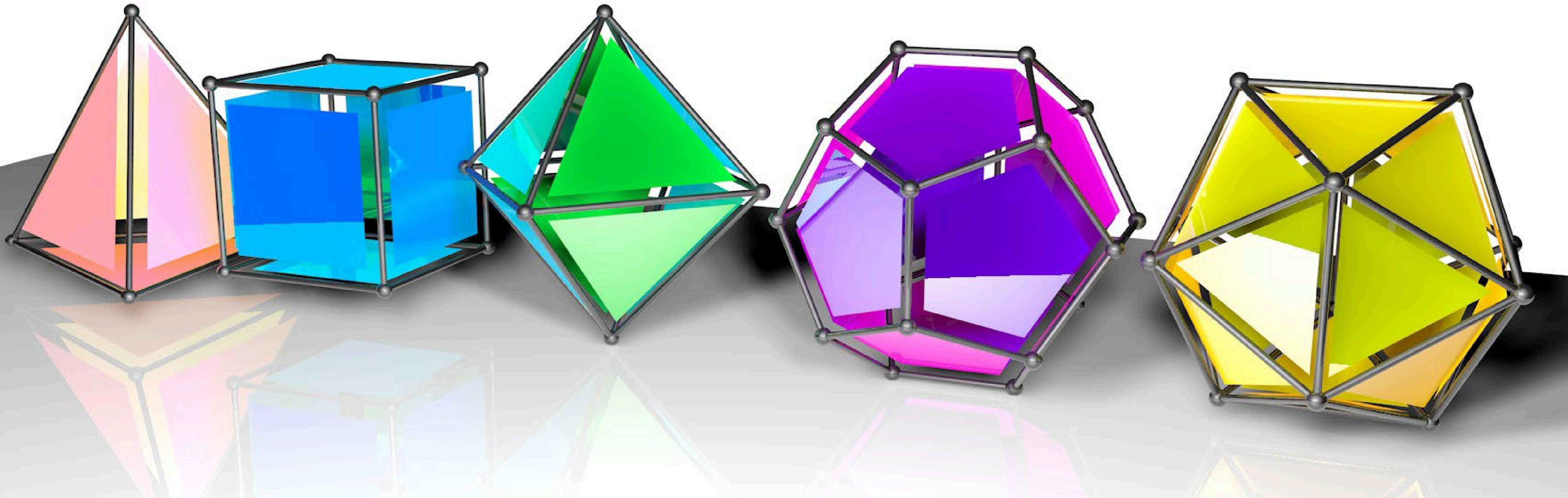
χ = [illegible]



$$\chi = \sum_k (-1)^k \dim C_k = \sum_k (-1)^k \dim H_c^k$$

THIS, LIKE COUNTING, IS A VALUATION…

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$



THIS, LIKE COUNTING, IS A VALUATION...

EULER INTEGRATION

MEASURABLE SETS = semi-algebraic sets (constructible)

MEASURABLE FUNCTIONS = constructible functions to $\mathbb{Z}$

EULER INTEGRAL = homomorphism

$$\int_X \_\_ \, d\chi \ : \ CF(X) \to \mathbb{Z} \qquad \int_X \mathbb{I}_A \, d\chi = \chi(A)$$

SCHAPIRA : VIRO : BARYSHNIKOV-GHRIST

WHY IS THIS AN INTEGRAL?

EULER INTEGRATION

FUBINI THEOREM : multiplicative nature of χ

INTEGRAL TRANSFORMS : convolution, Fourier(-Sato), Radon, …

(see work of Schapira +)

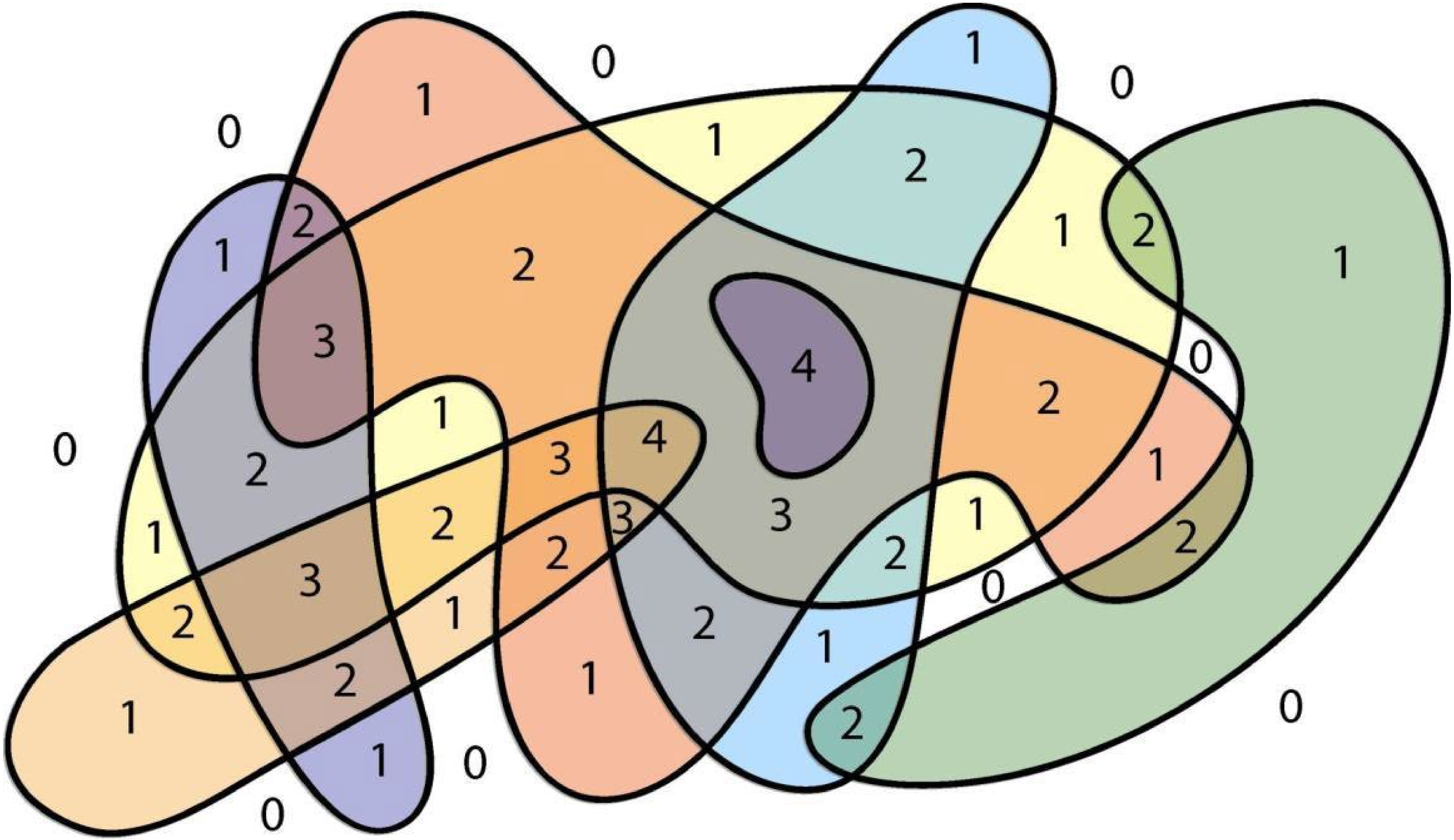
INTEGRAL GEOMETRY : Gauss-Bonnet for semi-algebraic sets

(see work of Brocker-Kuppe)

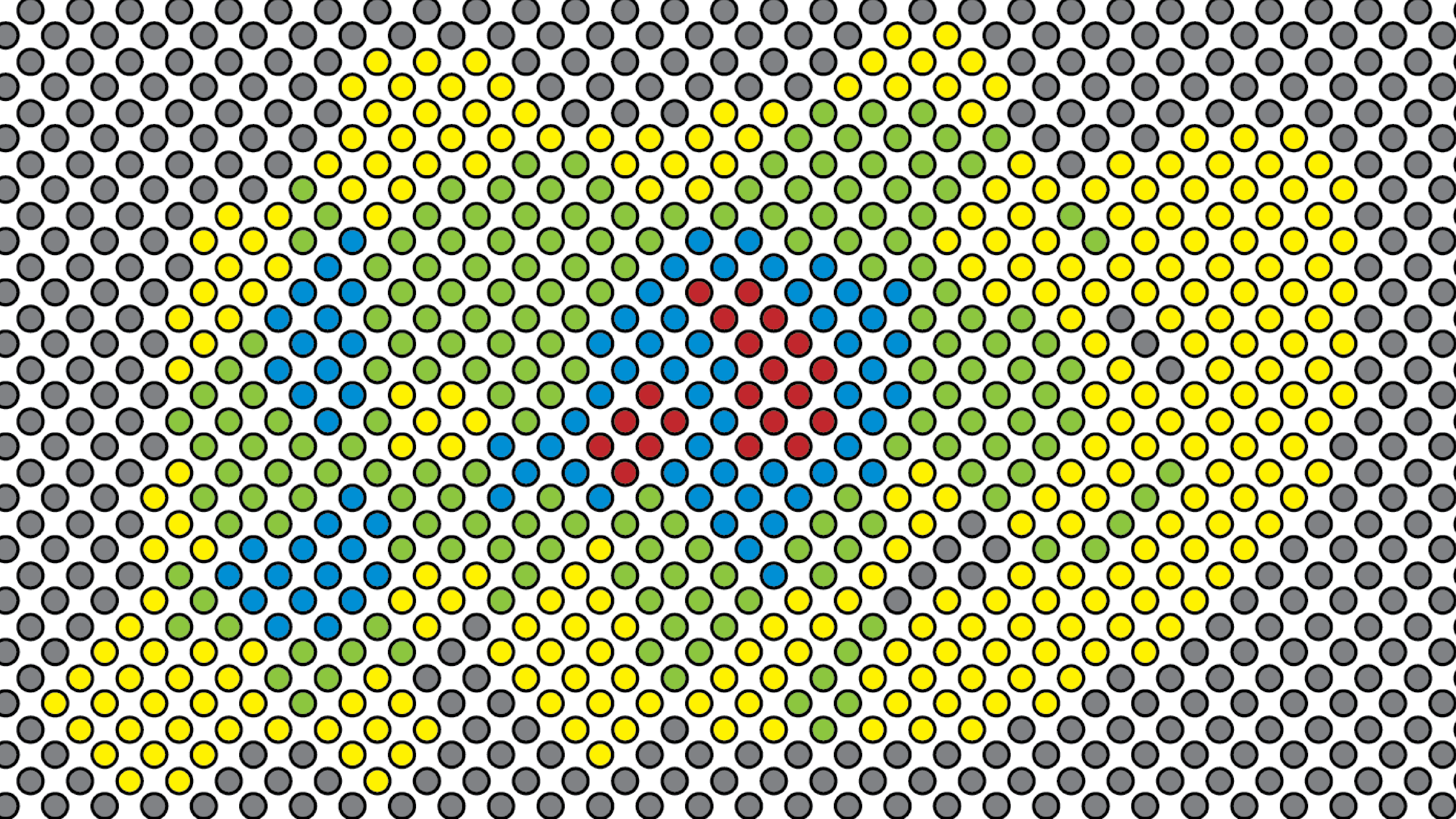
SIGNAL PROCESSING : lots of applications!

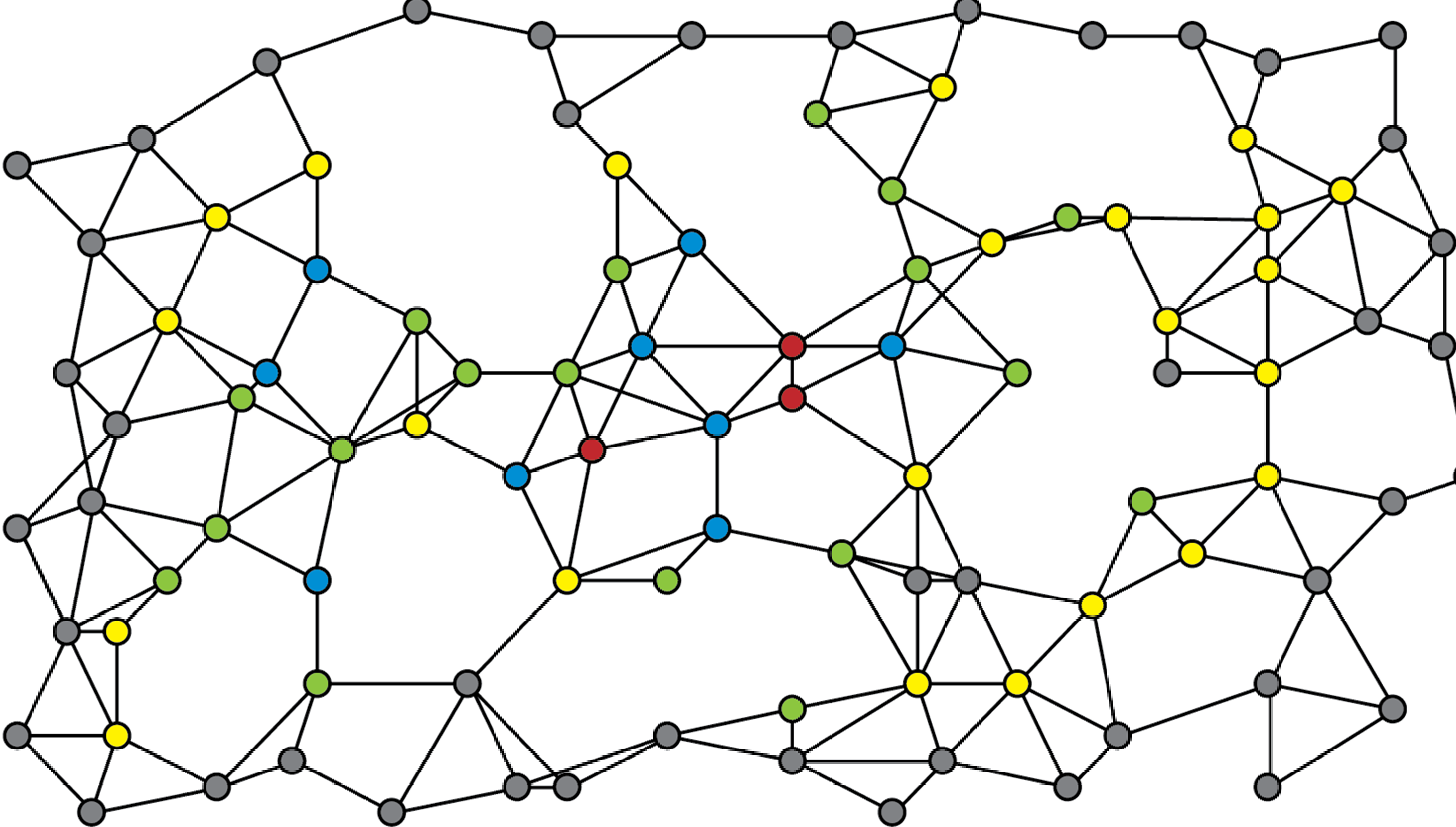
(see work of Baryshnikov-G)


Mineswe...
Game
Help



1
0
0
1
0
1
2
2
1
2
1
2
1
3
0
4
2
1
4
0
3
1
2
3
3
1
2
1
2
2
2
3
0
1
2
2
1
2
1
0
1
2
1
0
0





WHY DOES THIS ALL WORK?

ANSWER 1 : PROOFS!

THEOREM : Let $h \in CF(X)$ be a finite sum of indicator functions

$$h = \sum_{i=1}^{N} \mathbb{I}_{U_i} \quad : \quad \chi(U_i) = C$$ constant

then $$N = \frac{1}{C}\int_X h \, d\chi$$ depends only on h, not on the sum

YOU CAN PROVE THIS RIGHT NOW...

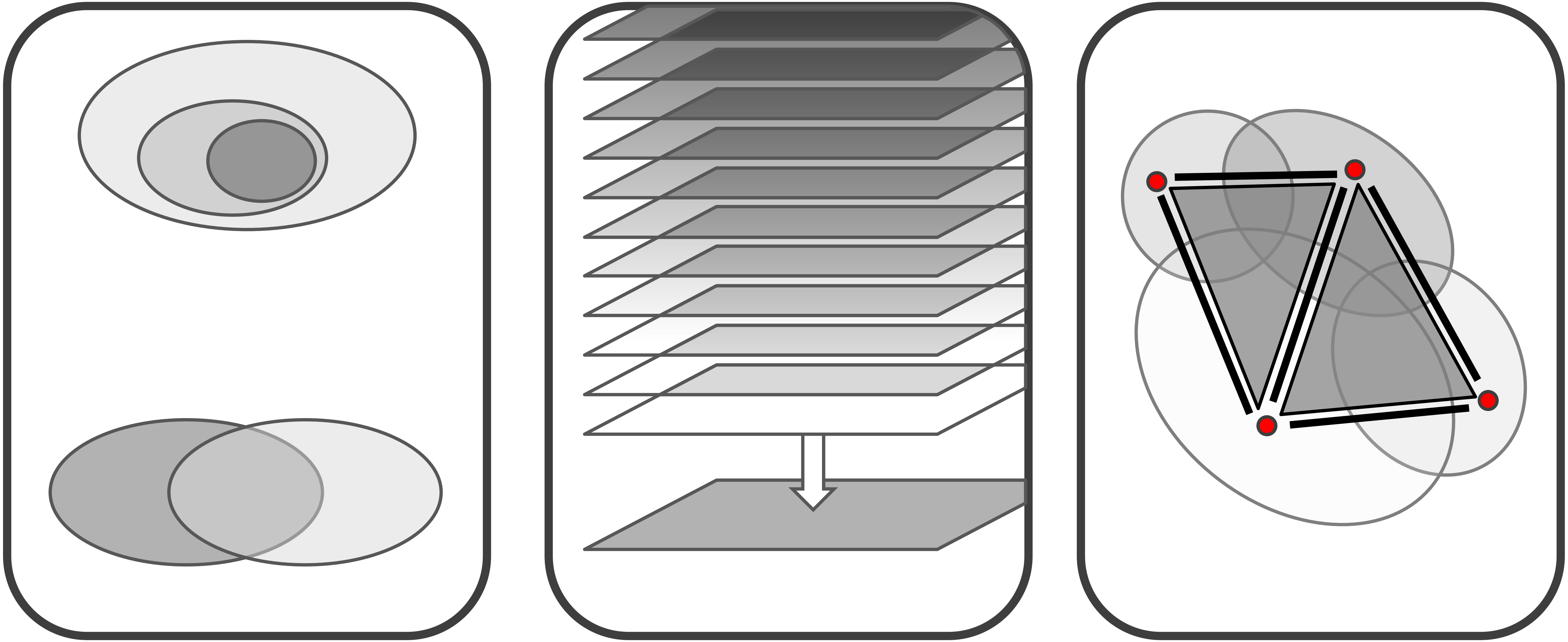
ANSWER 2 : THE POINT

ALL OF THE RESULTS IN EULER CALCULUS WERE INSPIRED BY AND FLOW DIRECTLY FROM

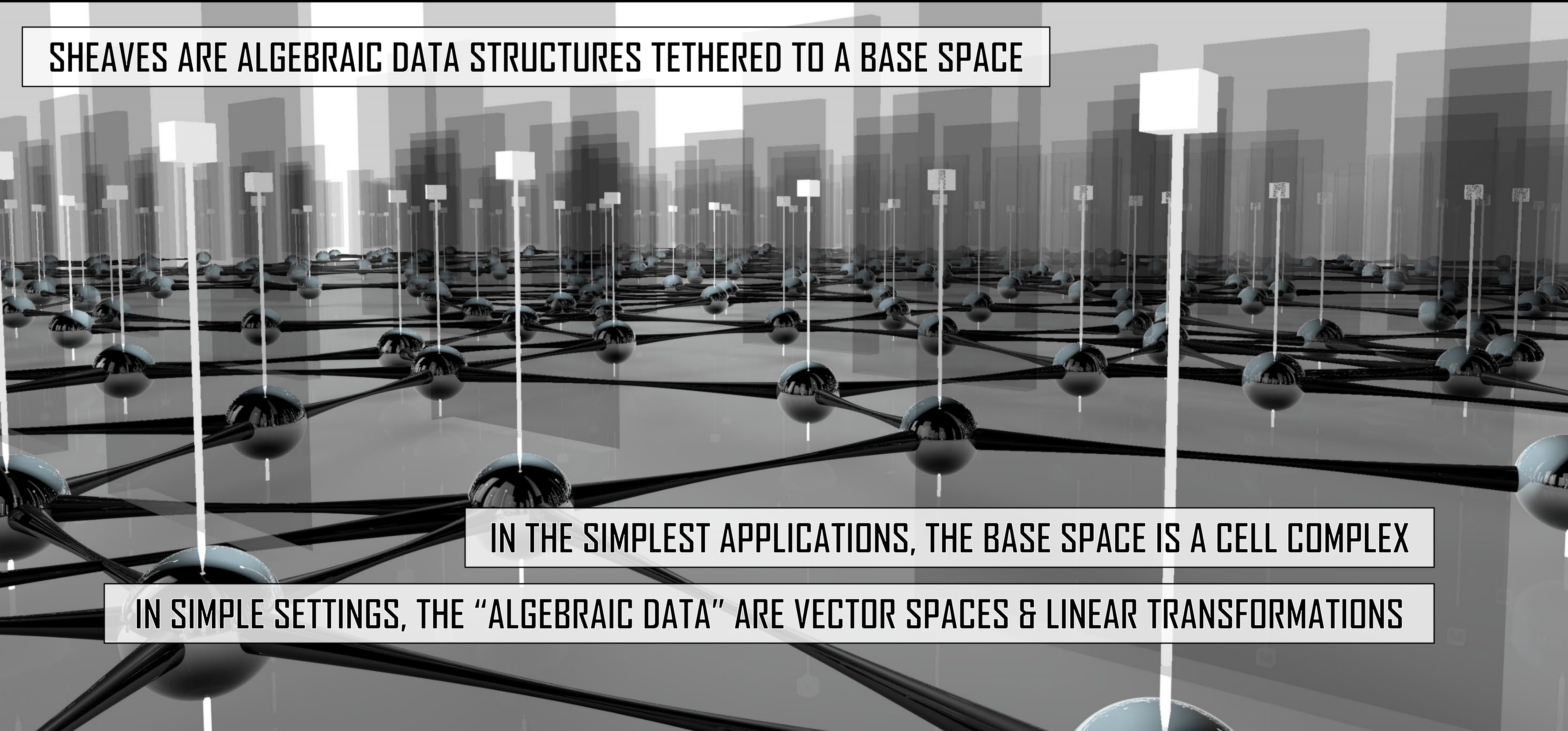
SHEAF THEORY

[SCHAPIRA < KASHIWARA ; VIRO < MACPHERSON]

SHEAVES



SHEAVES

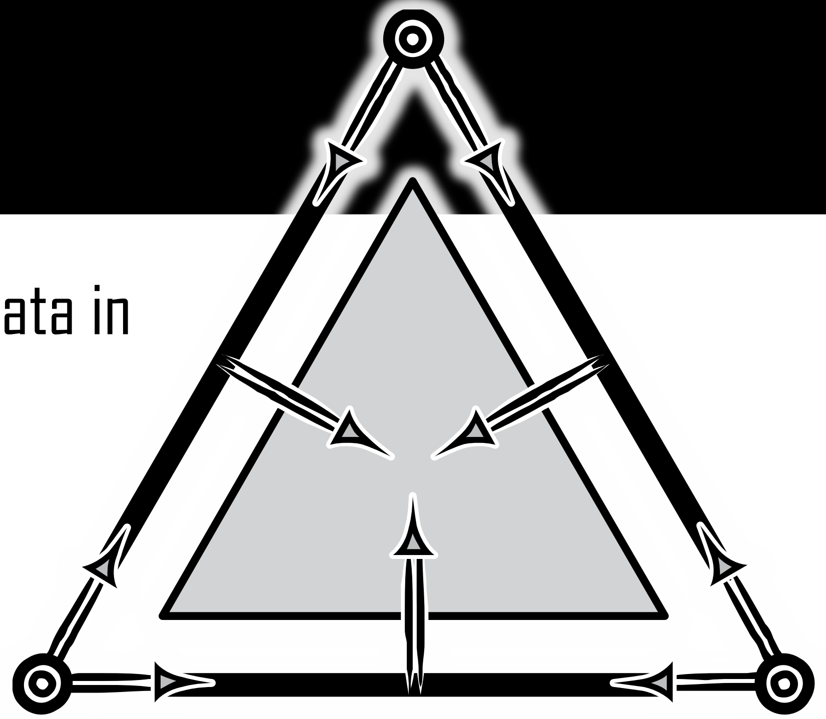


SHEAVES ARE ALGEBRAIC DATA STRUCTURES TETHERED TO A BASE SPACE

IN THE SIMPLEST APPLICATIONS, THE BASE SPACE IS A CELL COMPLEX

IN SIMPLE SETTINGS, THE “ALGEBRAIC DATA” ARE VECTOR SPACES & LINEAR TRANSFORMATIONS

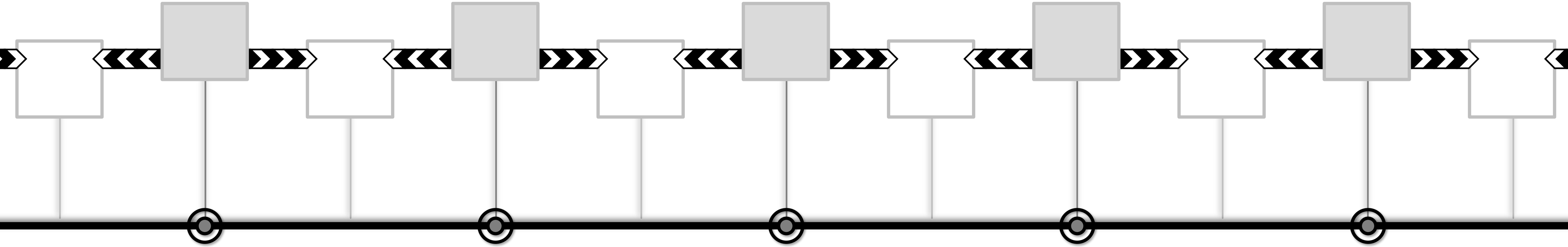
CELLULAR SHEAVES



DEFINITION: for X a CELL COMPLEX a (CELLULAR) **SHEAF** $\mathcal{F}$ with data in **C** is a FUNCTOR from the **FACE POSET** $(X, \trianglelefteq)$ to **C**

INTUITION: the **STALKS** $\mathcal{F}(\sigma)$ are DATA ; the (restriction) **MAPS** $\mathcal{F}(\sigma \trianglelefteq \tau)$ are COMPATIBILITY CONSTRAINTS

DEFINITION: a (CELLULAR) **COSHEAF** $\mathcal{F}$ is a CONTRAVARIANT SHEAF



CELLULAR SHEAVES

ARE IN FACT SHEAVES

& HAVE A RICH HISTORY

(BACLAWSKI, CURRY, KASHIWARA, LADKANI, MACPHERSON, SHEPERD, ZEEMAN)

CELLULAR SHEAVES

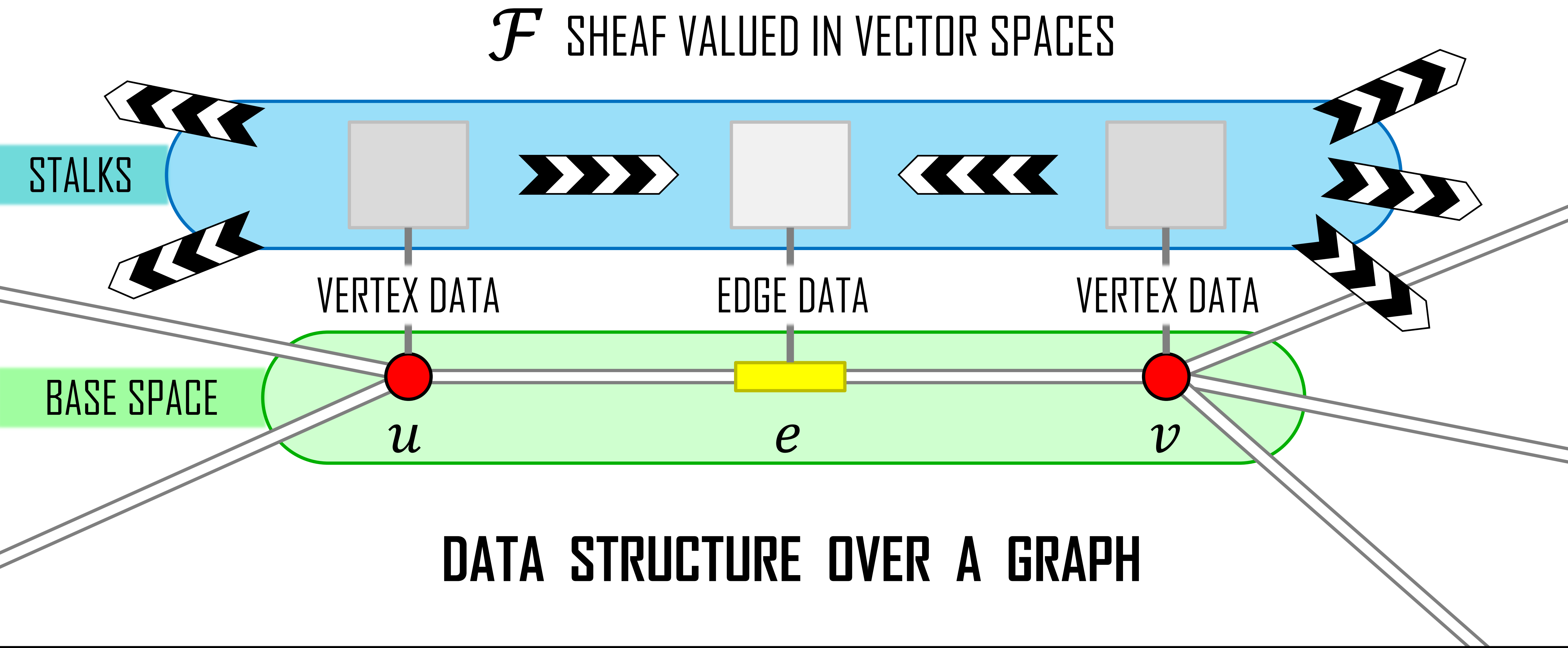
ARE IN FACT SHEAVES

IF YOU KNOW WHAT THE GLUING AXIOM
FOR A SHEAF OVER A TOPOLOGY REALLY MEANS…

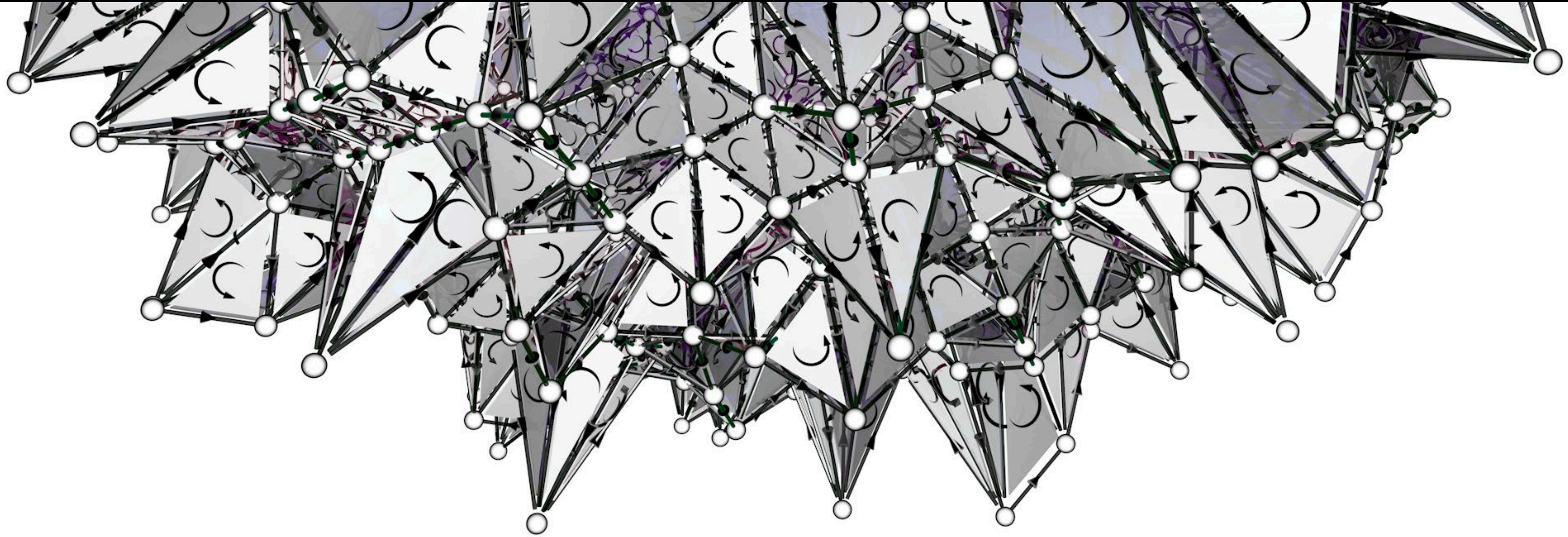
$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\mathcal{F}(U_k \subset U)} \prod_k \mathcal{F}(U_k) \xrightarrow{d} \prod_{i,j} \mathcal{F}(U_{ij})$$

ALL CELLULAR SHEAF APPROXIMATIONS MATCH

NETWORK SHEAVES



DATA STRUCTURE OVER A GRAPH



CELL COMPLEXES HAVE **HOMOLOGY**

HOW DID YOU LEARN HOMOLOGY? HOW DO YOU THINK ABOUT IT?

SHEAF COHOMOLOGY

DEFINITION: for a SHEAF $\mathcal{F}$ on X in **VECT** the COCHAIN COMPLEX of the sheaf, $C^\bullet(\mathcal{F})$, is the sequence of vector spaces that record DATA of $\mathcal{F}$ together with the COBOUNDARY MAP, d, comparing INCIDENT DATA

$$C^0(\mathcal{F}) \xrightarrow{d} C^1(\mathcal{F}) \xrightarrow{d} C^2(\mathcal{F}) \xrightarrow{d} C^3(\mathcal{F}) \xrightarrow{d}$$

VERTEX DATA | EDGE DATA | 2-CELL DATA | 3-CELL DATA

$$H^0(\mathcal{F}) \xrightarrow{0} H^1(\mathcal{F}) \xrightarrow{0} H^2(\mathcal{F}) \xrightarrow{0} H^3(\mathcal{F}) \xrightarrow{0}$$

DEFINITION: for a SHEAF $\mathcal{F}$ on X the **COHOMOLOGY** of the sheaf, $H^\bullet(\mathcal{F}) = \ker d \,/\mathrm{im}\, d$, is the sequence of quotient vector spaces that characterizes $\mathcal{F}$ on X. In particular, $H^0(\mathcal{F})$ = **GLOBAL SECTIONS** of the sheaf.

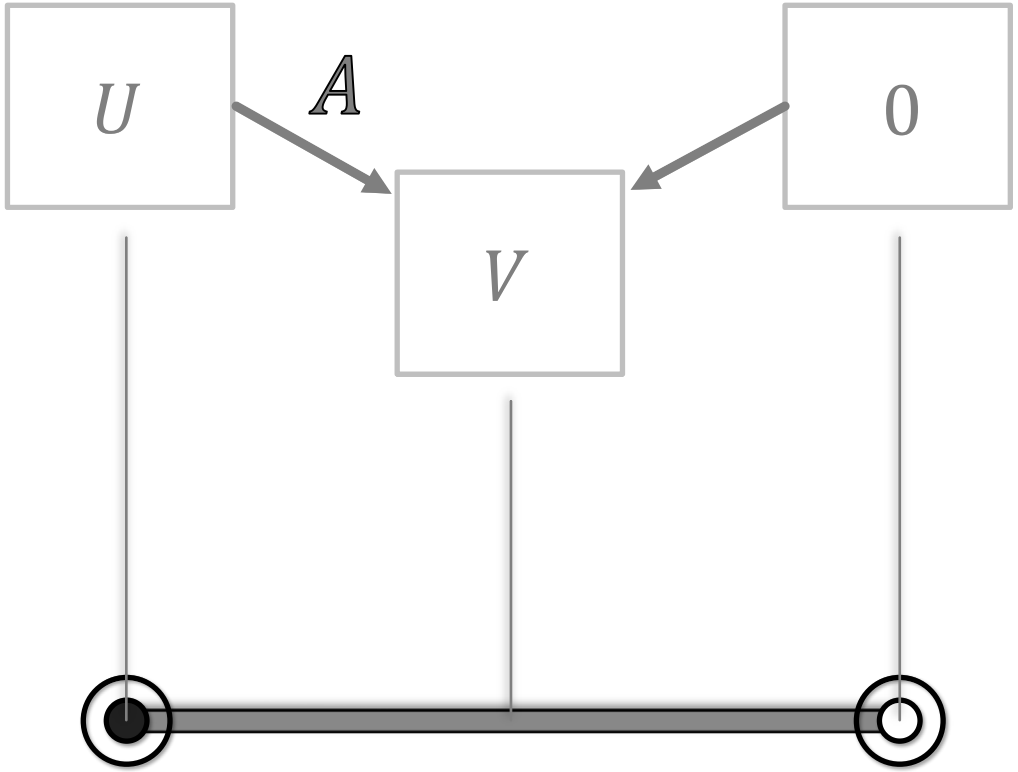
INTUITION: dimension of $H^0(\mathcal{F})$ is the number of independent consistent solutions to the constrained system

SHEAF COHOMOLOGY CLASSIFIES CONSISTENT SOLUTIONS & OBSTRUCTIONS THEREOF

[COSHEAVES POSSESS A DUAL HOMOLOGY THEORY…]

SHEAVES EXAMPLE

JUST A SIMPLE LINEAR TRANSFORMATION



$$H^0 = \ker A$$

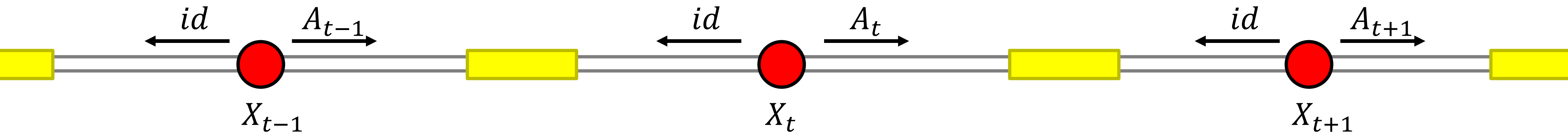
$$H^1 = \operatorname{coker} A$$

SHEAVES EXAMPLE

DISCRETE-TIME LINEAR SYSTEMS : all stalks are $\mathbb{R}^n$

Consider the typical system: $x_{t+1} = A_t x_t$

$\mathcal{F}$

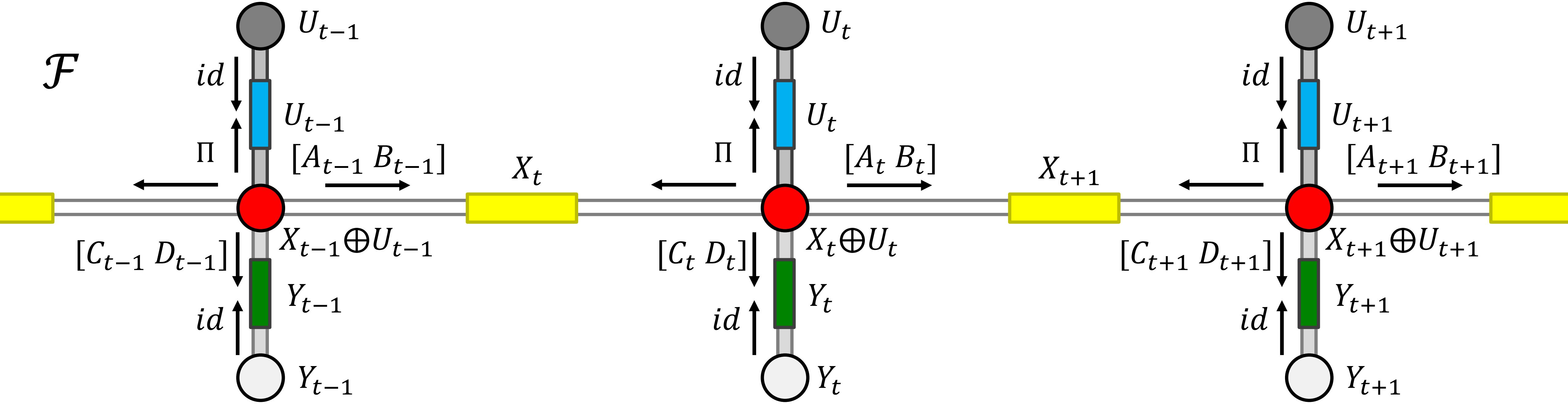


Global sections: $H^0\mathcal{F}$ classifies global-time solutions

DISCRETE-TIME LINEAR SYSTEMS WITH CONTROLS

J. Hansen

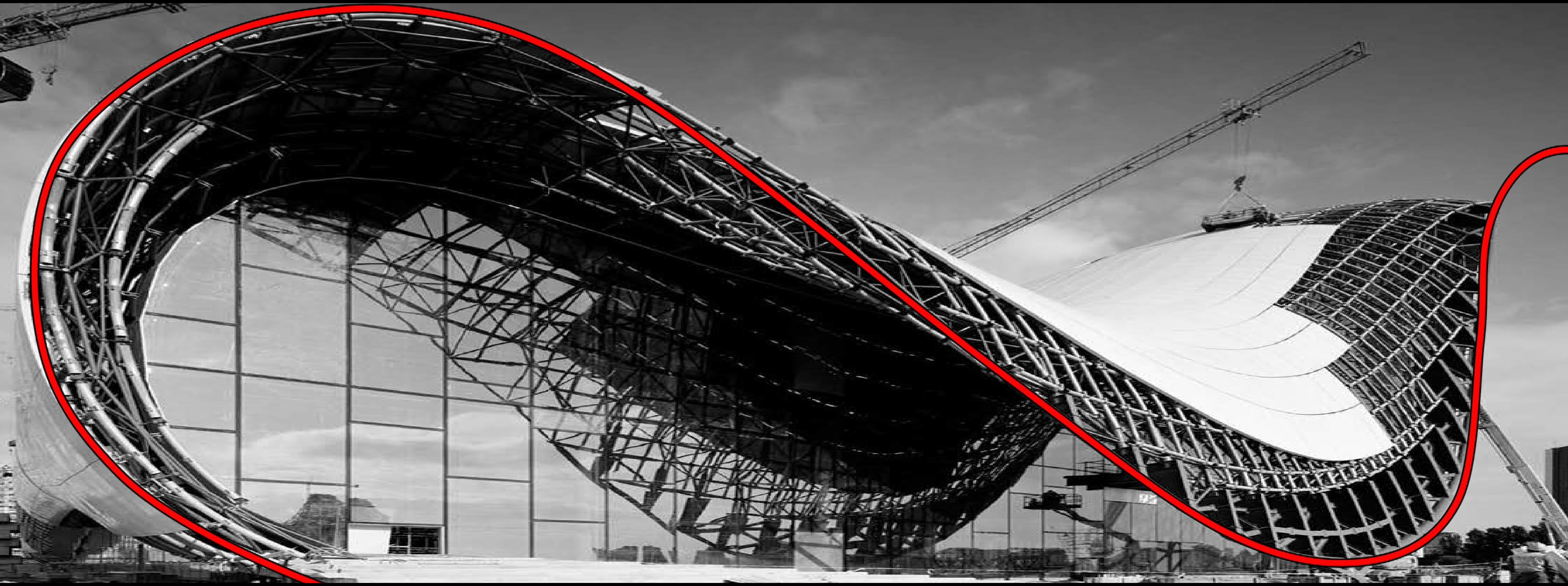
Consider the typical system: $x_{t+1} = A_t x_t + B_t u_t$; $y_t = C_t x_t + D_t u_t$



Global sections: $H^0\mathcal{F}$ classifies global-time solutions

SHEAVES EXAMPLE

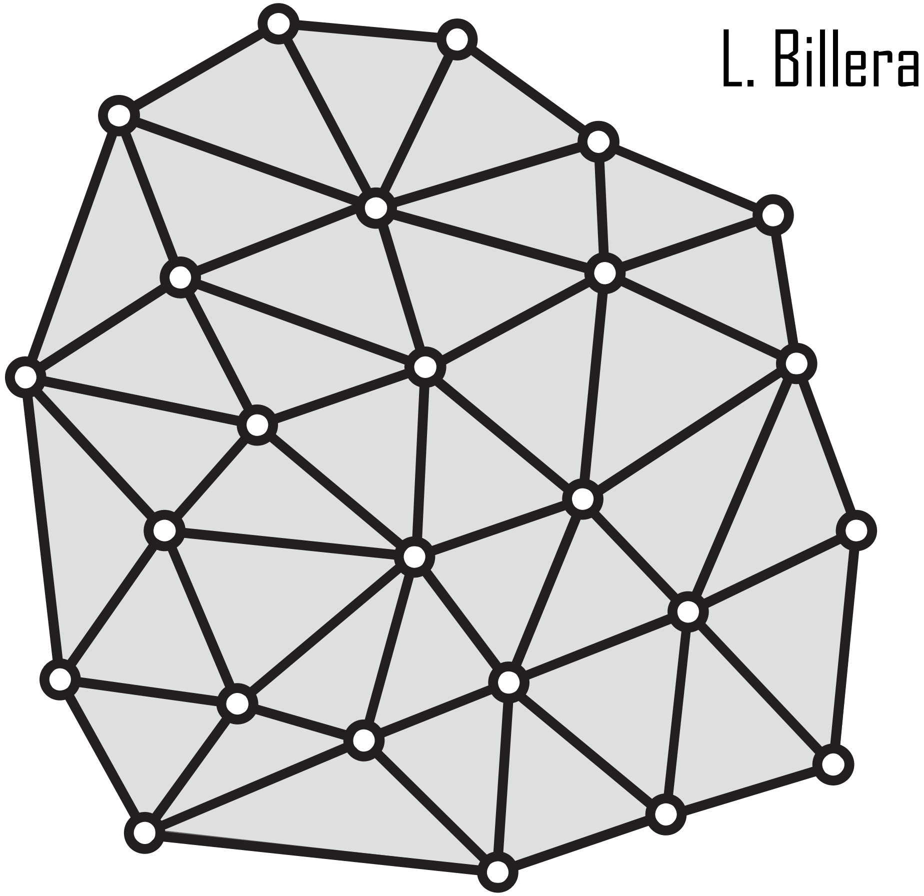
SPLINES

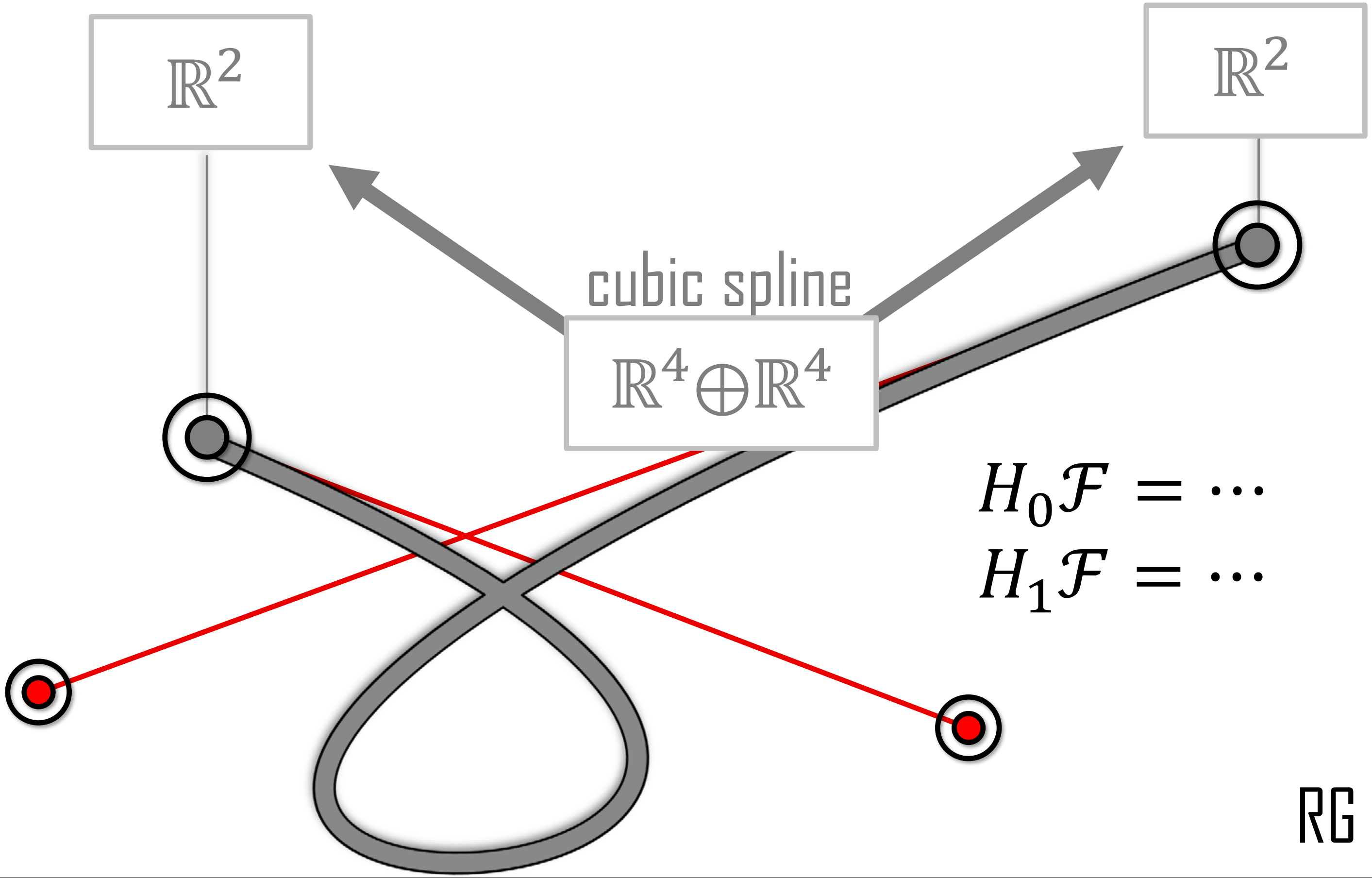


COSHEAVES EXAMPLE

SPLINES

BEZIER SPLINES





$H_0\mathcal{F} = \cdots$

$H_1\mathcal{F} = \cdots$

RG

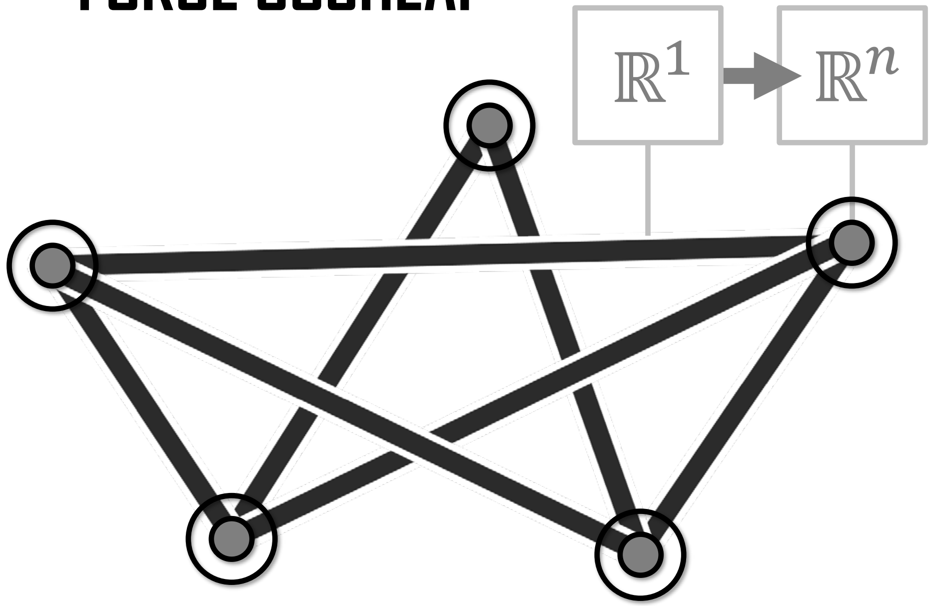
COSHEAVES EXAMPLE

GRAPHIC STATICS

w/Z. COOPERBAND

FORCE COSHEAF

[cf. CRAPO/WHITELY]



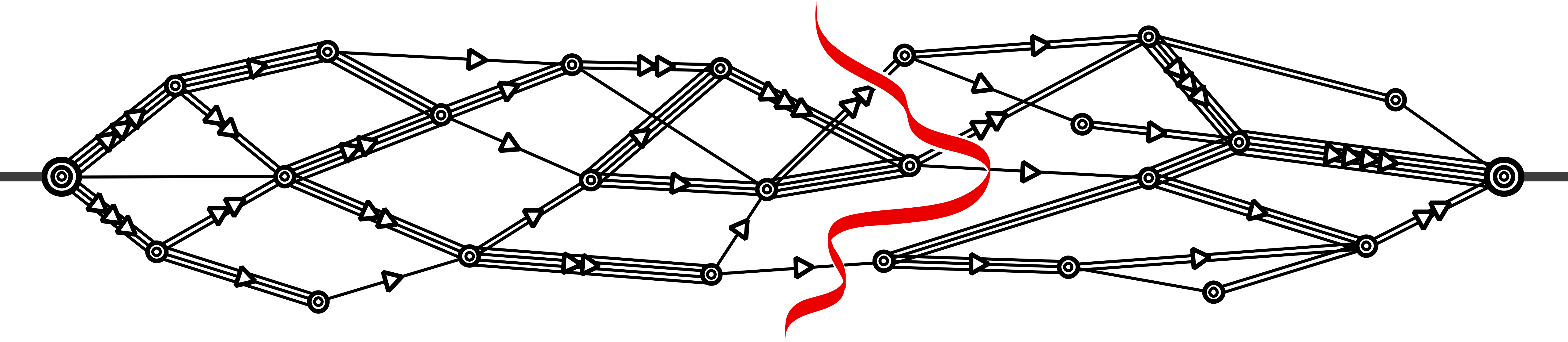
$C_0\mathcal{F}$ = STATIC LOADS on VERTICES

$C_1\mathcal{F}$ = AXIAL STRESSES on EDGES

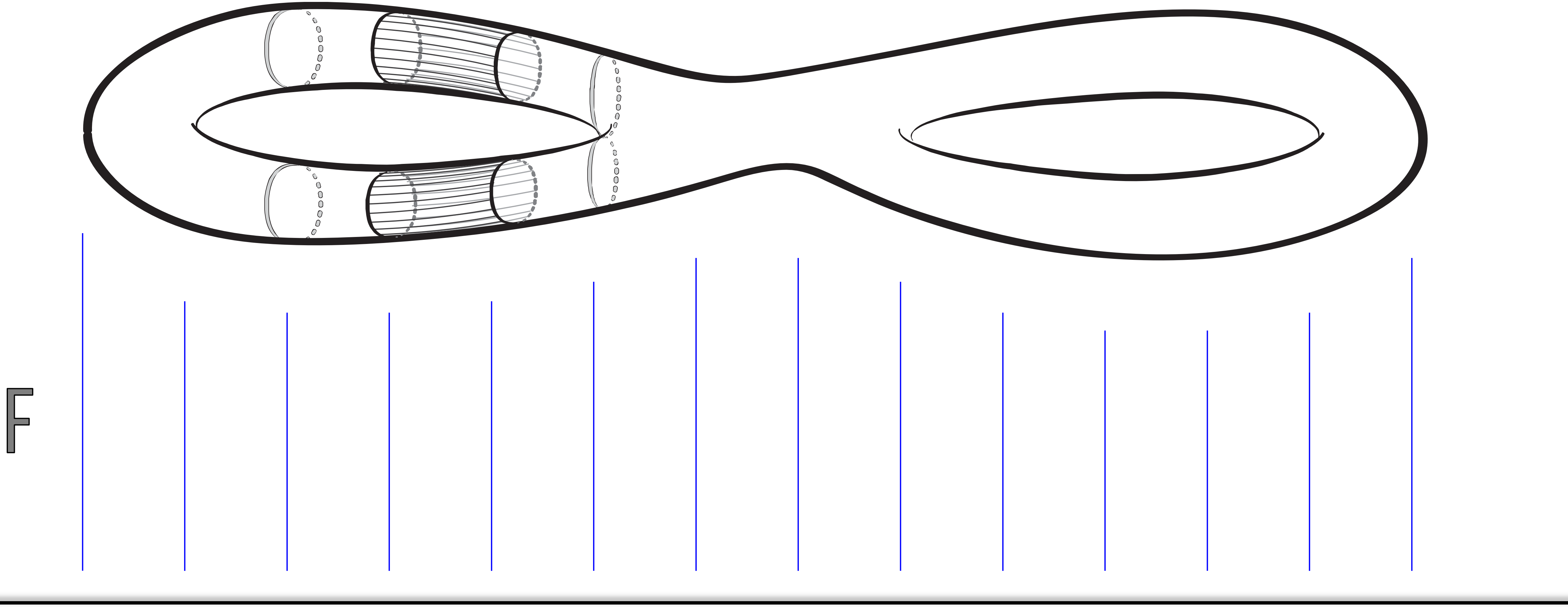
$H_1\mathcal{F}$ = SELF-STRESSES (NET ZERO FORCE)

$H_0\mathcal{F}$ = FLEXIBILITY/FREEDOM of TRUSS

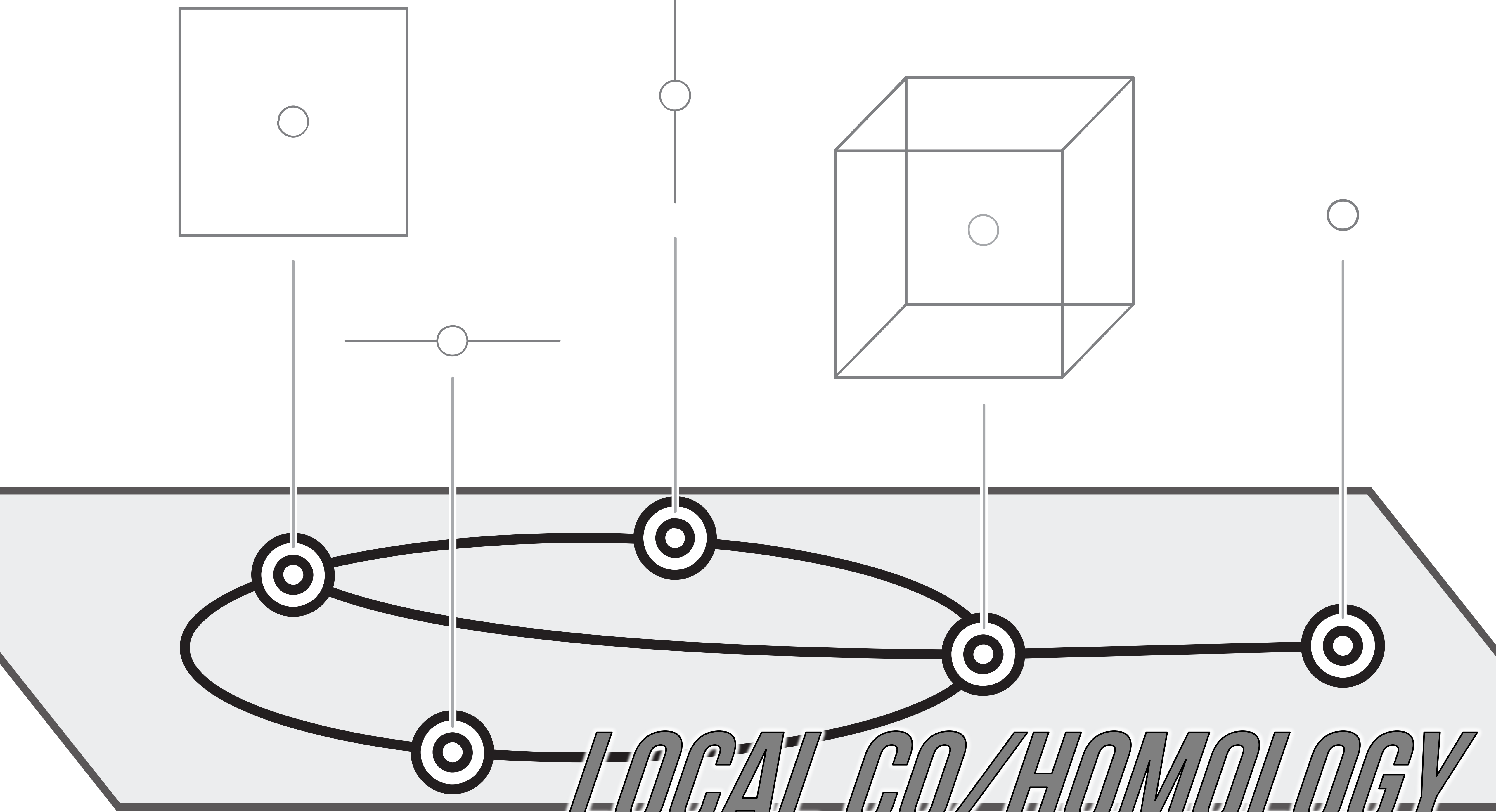
THEOREM [MAXWELL 1860s] : for a truss in $\mathbb{R}^n$: $\dim H_0\mathcal{F} = \dim H_1\mathcal{F} - \#E + n\#V$



NETWORK FLOW SHEAVES



LERAY FIBER SHEAF/COSHEAF

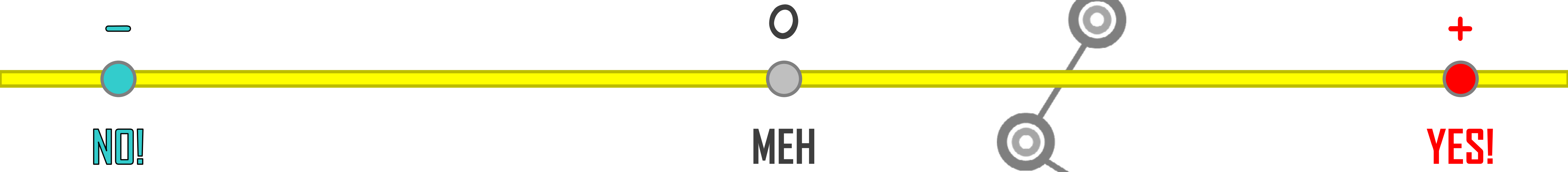


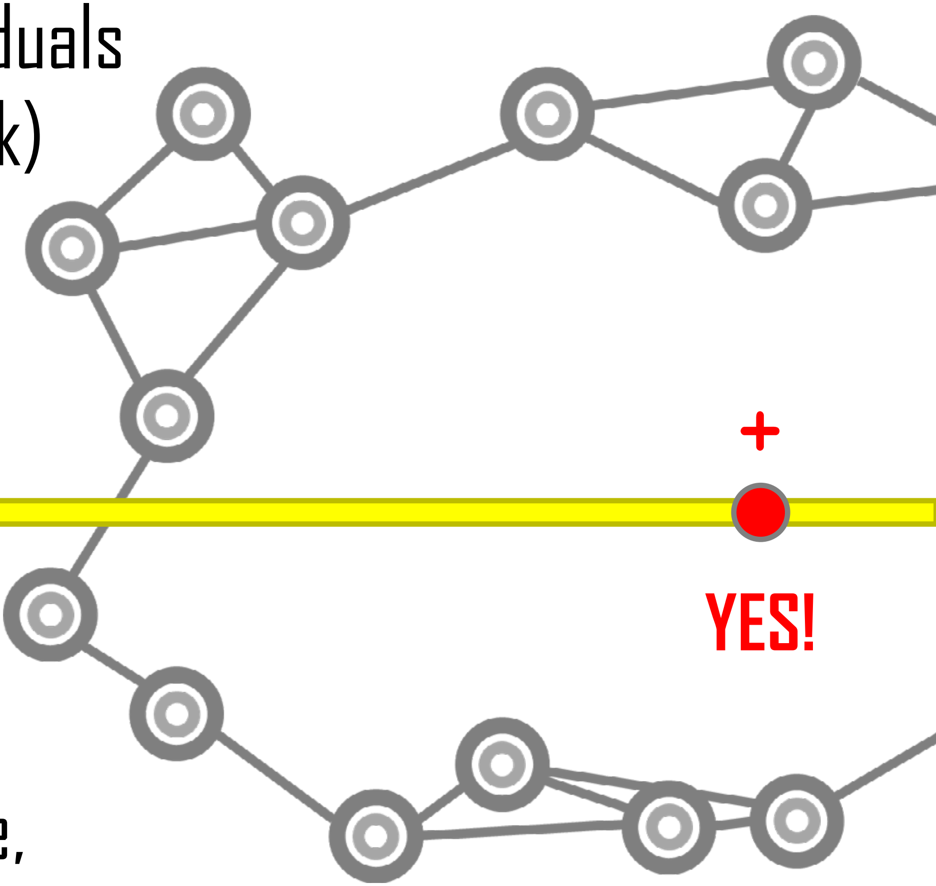
LOCAL CO/HOMOLOGY

DETAILED EXAMPLE : OPINIONS

OPINION DYNAMICS

CLASSIC : Let's say you have a social network of individuals
Model the network as an undirected graph (cf. FaceBook)
Assume that everybody has an opinion on some topic...





PROBLEM: What happens to peoples' opinions over time, assuming mutual influence?

CURRENT WORK

MOST CURRENT WORK HAS FOCUSED ON MORE COMPLEX SETTINGS / OUTCOMES

MULTIPLE OPINIONS: n-dimensional data over graph...

BOUNDED CONFIDENCE: influence only when opinions are close

HEGSELMAN-KRAUSE-type model, e.g. : many others

& SO MUCH MORE...

PREFERENCE FALSIFICATION and ANTAGONISTIC INTERACTION and SUSCEPTIBILITY and...

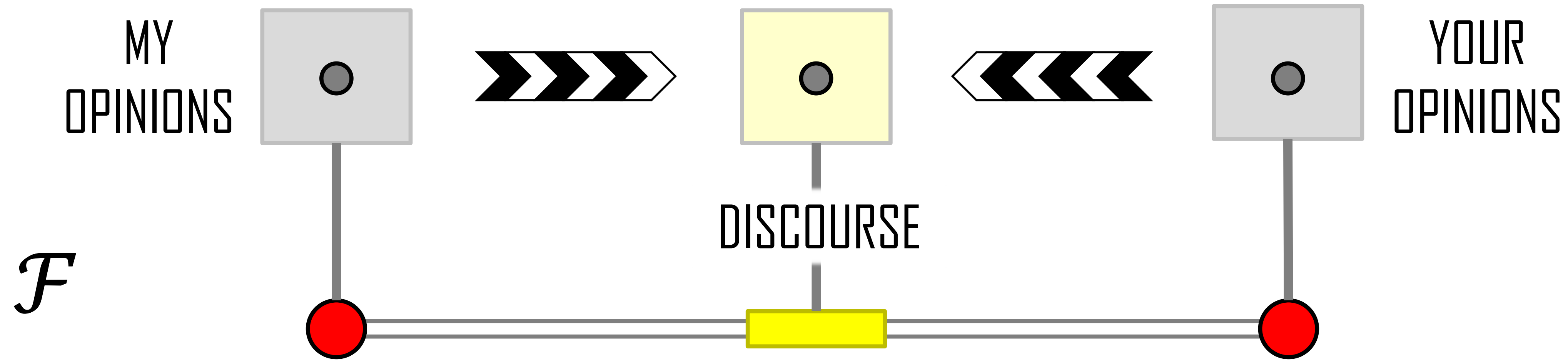
H. AHN ET AL [2020]
M. ALEXANIAN AND D. MCNAMARA [2018]
C. ALTAFINI [2012, 2013]
V. AMELKIN ET AL [2017]
T. ANTAL ET AL [2005, 2006]
H. BROOKS AND M. PORTER [2020]
C. CASTELLANO ET AL [2009, 2011, 2012]
A. JADBABAIE ET AL [2021]
H. NOORAZAR ET AL [2020]
A. PROSKURNIKOV ET AL [2020]
S. SCHWEIGHOFER ET AL [2020]
M. YE ET AL [2018, 2019]

Techniques in this subject include PROBABILITY, GRAPH THEORY, ODEs/PDEs, & more...

DISCOURSE SHEAVES

J. Hansen + G

CONSIDER THE FOLLOWING MODEL ON A SOCIAL NETWORK

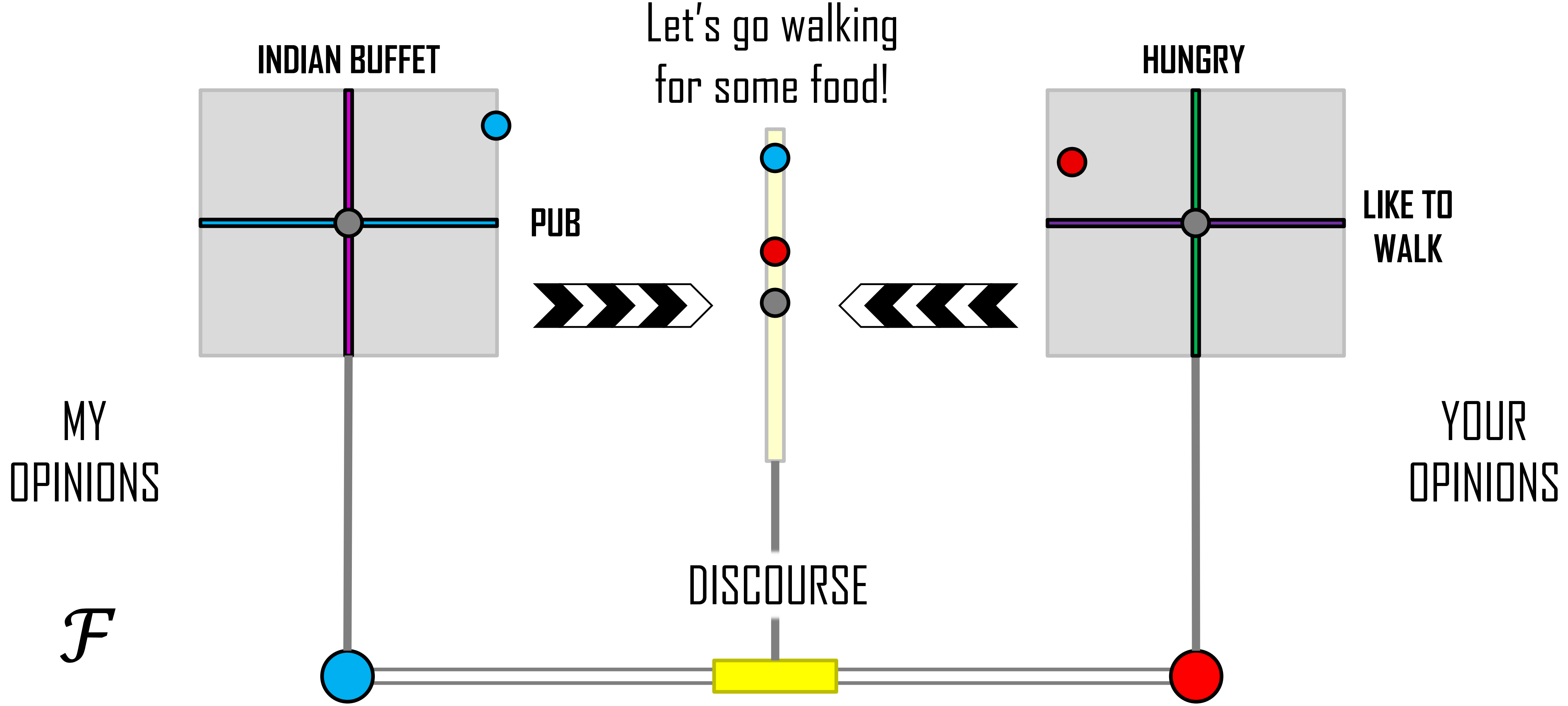


VERTEX STALKS : **OPINION SPACES** : private "basis" opinions from which policies are formulated

EDGE STALKS : **DISCOURSE SPACES** : public "basis" topics on which opinions are expressed

VERTEX-EDGE MAPS: **EXPRESSIONS** : how individuals choose to formulate opinions from bases

DISCOURSE SHEAVES EXAMPLE



LET'S THINK! THIS IS A VERY FLEXIBLE MODEL

AND IT ILLUSTRATES NETWORK SHEAVES VERY WELL...

COHOMOLOGY & OPINION DISTRIBUTIONS

$C^0(\mathcal{F})$	0 - COCHAINS	opinion distributions (private)
$C^1(\mathcal{F})$	1 - COCHAINS	pairwise discussions (public)
$d: C^0 \rightarrow C^1$	COBOUNDARY	aggregate public disagreement
$H^0(\mathcal{F})$	GLOBAL SECTIONS	harmonic opinion distributions

WHAT MORE COULD WE DO WITH THIS?

THIS IS JUST THE BEGINNING

WITH A LOT OF VERY SIMPLE EXAMPLES TO GET USED TO THE MAIN IDEAS...

CELLULAR SHEAVES

THERE HAVE BEEN A LOT OF RECENT APPLICATIONS...

NETWORK CODING	HIRAOKA-G
TARGET ENUMERATION / TRACKING	BARYSHNIKOV-G ; KRISHNAN-G
REEB GRAPHS / SPACES	DESILVA-MUNCH-PATEL
PERSISTENT HOMOLOGY	CURRY ; KASHIWARA-SCHAPIRA ; MACPHERSON-PATEL
PERSISTENT HOMOLOGY TRANSFORM	CURRY-TURNER-MUKHERJEE
DISTRIBUTED OPTIMIZATION	HANSEN-G
LEARNING OPTIMAL STRATIFICATIONS	NANDA
GRAPH SIGNAL PROCESSING / ML	BODNAR-DIGIOVANNI+..+BRONSTEIN

MORE TOMORROW...