Unbounded families of minimal 2-surfaces and Einstein 4-manifolds

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Bounded families

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In our case, the bound can be imposed on an energy (the Morse index and area, or the $L^2$-norm of the Riemann curvature and volume). Then due to $\varepsilon$-regularity theorems, if this energy is bounded by $\leq A$ along a sequence, then the energy can concentrate at most at $\sim A$ points and the curvature can blow-up at most at $\sim A$ points. Then a bubbling argument enables to conclude finiteness.
Examples

Sharp, Chodosh-Ketover-Maximo:

If \( \{\Sigma_i\} \) is a sequence of minimal surfaces in \((N, g)\) such that Morse index(\(\Sigma_i\)) \(\leq C\), and Area(\(\Sigma_i\)) \(\leq C\) then the genus of \(\Sigma_i\) is uniformly bounded.
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Anderson, Bando-Kasue-Nakajima, Gao, Anderson-Cheeger:

If \( \{(M_i, g_i)\} \) is a sequence of Einstein 4-manifolds such that \(\chi(M_i) \leq C\) and \(\text{Vol}(M_i, g_i) \geq C^{-1}\) and \(\text{Diam}(M_i, g_i) \leq C\) then the number of diffeomorphism types of \(M_i\) is finite.
The previous results were improved and generalized by removing one of the geometric bounds. The general strategy remains the same: the bounds left still suffice to produce well-behaved limits, and the compactness/finiteness results follow from analysing those limits. The difficulty is to understand the nature of these limits (minimal laminations, or non-collapsed Einstein Ricci-limits).
X. Zhou-H. Li, Chodosh-Ketover-Maximo:

If \( \{ \Sigma_i \} \) is a sequence of minimal surfaces in \((N, g)\) such that Morse index\( (\Sigma_i) \leq C \) then \( \Sigma_i \) converges to a smooth minimal lamination and the topology can concentrate only at \( \lesssim C \) points.
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Cheeger-Naber:

If \(\{(M_i, g_i)\}\) is a sequence of Einstein 4-manifolds such that \(\text{Vol}(M_i, g_i) \geq C^{-1}\) and \(\text{Diam}(M_i, g_i) \leq C\) then the number of diffeomorphism types of \(M_i\) is finite.
What about unbounded families?

In general, we would like to say something about minimal surfaces or Einstein manifolds without a priori bounds (such families exist!). One possibility is to try to relate quantitatively geometric, analytic, and topological invariants. For minimal surfaces: how are area, Morse index, genus related? For Einstein manifolds: how are “minimal volume” and Euler characteristic related?

\[
\text{minvol}(M) := \inf \left\{ \text{Vol}(M, g) \mid |\text{sec}_g| \leq 1 \right\}
\]

(Gromov)
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In general, we would like to say something about minimal surfaces or Einstein manifolds without a priori bounds (such families exist!). One possibility is to try to relate quantitatively geometric, analytic, and topological invariants. For minimal surfaces: how are area, Morse index, genus related? For Einstein manifolds: how are “minimal volume” and Euler characteristic related?

That is difficult due to the fact that taking limit is either useless or not well-defined. As a first step towards such unbounded estimates, we can rely on the large/small decomposition principle. The object is divided into two pieces A and B: the first piece A is locally trivial but globally controlled, while the other piece B is locally special. Our goal is then to quantitatively control the piece A.
Consider $v_0 > 0$ and $\epsilon > 0$ small constants. For an Einstein 4-manifold $(M, g)$, define at $p \in M$:

$$r_\epsilon(p) := \sup\{r \in (0, 1]; \int_{B(p, r)} |\text{Rm}|^2 \leq \epsilon\},$$

and set

"thick" $M_{> v_0} := \{x \in M; \quad \text{Vol}(B(x, r_\epsilon(x))) > v_0 r_\epsilon(x)^4\}$,

"thin" $M_{\leq v_0} := \{x \in M; \quad \text{Vol}(B(x, r_\epsilon(x))) \leq v_0 r_\epsilon(x)^4\}$. 
$M_{>r_0}$ is locally trivial by e.g.

$M_{<r_0}$ carry a "F-structure"

( Cheeger - Gromov)
Let \((N, g)\) be a 3-manifold. Consider \(n_0 > 0\) a large constant and \(\bar{r} > 0\) a small constant. For a minimal surface \(\Sigma \subset N\), define at \(p \in N\): 

\[
\Sigma(p) := \sup\{r \leq \bar{r}; \quad \Sigma \cap B(p, r) \text{ is stable}\},
\]

and set

\[
\Sigma_{> n_0} := \{x \in \Sigma; \quad \Sigma \cap B(x, s(x)) \text{ has area larger than } n_0 s(x)^2\},
\]

\[
\Sigma_{\leq n_0} := \{x \in \Sigma; \quad \Sigma \cap B(x, s(x)) \text{ has area at most } n_0 s(x)^2\}.
\]
here non-sheeted part $\Sigma_{\neq n_0}$ is locally simple

sheeted part $\Sigma_{> n_0}$
Theorem 1: Let \((N, g)\) be a closed 3-manifold, there is a constant 
\(C = C(N, g)\) such that for any minimal surface \(\Sigma \subset N\) and \(n_0 > 0\), 
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genus(\Sigma_{\leq n_0}) \leq Cn_0(\text{Morse index}(\Sigma) + 1).
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Theorem 2: There are constants \(C, \epsilon, v_0\), such that for any closed Einstein 4-manifold \((M, g)\), \(M_{> v_0}\) admits a metric \(h\) with

\[
|\text{Rm}_h| \leq 1, \quad \text{injrad}_h \geq 1 \quad \text{and} \quad \text{Vol}(M_{\geq v_0}, h) \leq C\chi(M).
\]
Results expressed with triangulation

Theorem 1: Let \((N, g)\) be a closed 3-manifold, there is a constant \(C = C(N, g)\) such that for any minimal surface \(\Sigma \subset N\) and \(n_0 > 0\), \(\Sigma_{\leq n_0}\) has a triangulation with degree \(\leq C\) and total number of vertices \(\leq Cn_0(\text{Morse index}(\Sigma) + 1)\).

Theorem 2: There are constants \(C, \varepsilon, \nu_0\), such that for any closed Einstein 4-manifold \((M, g)\), \(M_{> \nu_0}\) has a triangulation with degree \(\leq C\) and total number of vertices \(\leq C\chi(M)\).
Proof idea for minimal surfaces

Fix $n_0 \gg 1$. To simplify the discussion, we start with a minimal surface $\Sigma \subset N$ which has area $\ll n_0$ (so that $\Sigma \ll n_0 = \Sigma$) and $\Sigma$ is not stable in any ball of radius $\geq \bar{r}$ (depending only on the ambient 3-manifold $N$).
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Goal: construct a triangulation of $\Sigma$ with degree $\leq C$ and total number of vertices less than $\leq C$.Morse index($\Sigma$), for $C > 0$ independent of $\Sigma$. 
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By the $\epsilon$-regularity theorem, if in the ball $B(p, r) \subset N$, the minimal surface is stable, then the second fundamental form of $\Sigma$ is pointwise bounded in $B(p, r/2)$. Hence $\Sigma \cap B(p, r/2)$ has bounded topology. To simplify the discussion, we will assume that $\Sigma \cap B(p, r/2)$ is a flat disk of scale $\sim r/2$. 

\begin{center}
\includegraphics[width=0.5\textwidth]{proof_idea.png}
\end{center}
Proof idea for minimal surfaces

- At each \( p \in \Sigma \), remember the “stability radius”

\[
s(p) := \sup\{ r \leq \bar{r} ; \quad \Sigma \cap B(p, r) \text{ is stable} \},
\]

so that \( \Sigma \cap B(p, s(p)) \) is stable but \( \Sigma \cap B(p, 2s(p)) \) has Morse index \( \geq 1 \).

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Remember we are assuming that $\Sigma \cap B(p, s(p)/2)$ is a disk of size $\approx s(p)/2$.

- Cover $\Sigma$ with $q_1$ balls of the form $B(p, s(p)/2)$, and $q_1$ as small as possible. We can construct a triangulation of $\Sigma$ with $\lesssim q_1$ vertices.
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- Find a subfamily of disjoint \( q_2 \) balls, where \( q_2 \geq q_1 \).
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- Find a subfamily of disjoint $q_2$ balls, where $q_2 \gtrsim q_1$.

We would be done if we could prove that Morse index$(\Sigma) \gtrsim q_2$. But the problem is that the balls in the subfamily are not disjoint enough.
\[ \text{ind}(\mathcal{Z}) \geq 2 \]
Proof idea for minimal surfaces

\[ \sum \setminus \{ \text{almost conical regions} \} \]

In fact, the issue can occur only around almost conical regions. But almost conical regions do not add topology, so we can remove these regions: we cover the complement of these regions by balls \( B(x_1, s(x_1)/2), \ldots, B(x_{q_1}, s(x_{q_1})/2) \) then try to get a subfamily of balls \( B(y_1, s(y_1)/2), \ldots, B(y_{q_2}, s(y_{q_2})/2) \), so that

\[ q_2 \geq q_1 \text{ and } \forall i \neq j, B(y_i, 2s(y_j)) \cap B(y_j, 2s(y_j)) = \emptyset. \]

(This is possible by a counting argument.)

\[ \Rightarrow \text{ Morse index } \geq q_2 \gg q_1 \]
Idea of proof for Einstein 4-manifolds

The previous combinatorial arguments work well for Einstein $n$-manifolds with $\int |\mathcal{R}m|^{n/2}$ replacing the Morse index. However in dimension 4, there is a more efficient way to proceed by using Cheeger-Naber.
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For simplicity we will assume that $(M, g)$ is an Einstein 4-manifold, non-collapsed: $\text{Vol}(B(p, 1)) > v_0 > 0$ for any $p \in M$. $\Rightarrow M_{>v_0}=M$

Goal: construct a triangulation of $M$ with degree $\leq C$ and total number of vertices less than $\leq C\cdot \chi(M)$, for $C > 0$ independent of $M$. 

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- Conclude.