Unbounded families of minimal 2-surfaces and Einstein 4-manifolds

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In our case, the bound can be imposed on an energy (the Morse index and area, or the L^2-norm of the Riemann curvature and volume). Then due to ϵ-regularity theorems, if this energy is bounded by $\leq A$ along a sequence, then the energy can concentrate at most at $\lesssim A$ points and the curvature can blow-up at most at $\lesssim A$ points. Then a bubbling argument enables to conclude finiteness.

Examples

Sharp, Chodosh-Ketover-Maximo:

If $\{\Sigma_i\}$ is a sequence of minimal surfaces in (N, g) such that Morse index$(\Sigma_i) \leq C$, and Area$(\Sigma_i) \leq C$ then the genus of Σ_i is uniformly bounded.

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Anderson, Bando-Kasue-Nakajima, Gao, Anderson-Cheeger:

If $\{(M_i, g_i)\}$ is a sequence of Einstein 4-manifolds such that $\chi(M_i) \leq C$ and Vol$(M_i, g_i) \geq C^{-1}$ and Diam$(M_i, g_i) \leq C$ then the number of diffeomorphism types of M_i is finite.

Improvements by removing one of the bounds

The previous results were improved and generalized by removing one of the geometric bounds. The general strategy remains the same: the bounds left still suffice to produce well-behaved limits, and the compactness/finiteness results follow from analysing those limits. The difficulty is to understand the nature of these limits (minimal laminations, or non-collapsed Einstein Ricci-limits).

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Cheeger-Naber:

If $\{(M_i, g_i)\}$ is a sequence of Einstein 4-manifolds such that Vol$(M_i, g_i) \geq C^{-1}$ and Diam$(M_i, g_i) \leq C$ then the number of diffeomorphism types of M_i is finite.

~~$\chi(M_i) < C$~~

What about unbounded families?

In general, we would like to say something about minimal surfaces or Einstein manifolds without a priori bounds (such families exist!). One possibility is to try to relate quantitatively geometric, analytic, and topological invariants. For minimal surfaces: how are area, Morse index, genus related? For Einstein manifolds: how are "minimal volume" and Euler charateristic related?

(Gromov)

$$\mathrm{minvol}(M) := \inf \{ \mathrm{Vol}(M,g);\ |\mathrm{sec}_g| \leq 1 \}$$

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In general, we would like to say something about minimal surfaces or Einstein manifolds without a priori bounds (such families exist!). One possibility is to try to relate quantitatively geometric, analytic, and topological invariants. For minimal surfaces: how are area, Morse index, genus related? For Einstein manifolds: how are "minimal volume" and Euler charateristic related?

That is difficult due to the fact that taking limit is either useless or not well-defined. As a first step towards such unbounded estimates, we can rely on the large/small decomposition principle. The object is divided into two pieces A and B: the first piece A is locally trivial but globally controlled, while the other piece B is locally special. Our goal is then to quantitatively control the piece A.

Thick/thin decomposition for Einstein 4-manifolds

Consider $v_0 > 0$ and $\epsilon > 0$ small constants. For an Einstein 4-manifold (M, g), define at $p \in M$:

$$r_\epsilon(p) := \sup\{r \in (0,1]; \quad \int_{B(p,r)} |\operatorname{Rm}|^2 \leq \epsilon\},$$

and set

"thick" $$M_{>v_0} := \{x \in M; \quad \operatorname{Vol}(B(x, r_\epsilon(x))) > v_0 r_\epsilon(x)^4\},$$

"thin" $$M_{\leq v_0} := \{x \in M; \quad \operatorname{Vol}(B(x, r_\epsilon(x))) \leq v_0 r_\epsilon(x)^4\}.$$

$M_{>v_0}$ is locally trivial by ε-reg.

$M_{\leq v_0}$ carries a "F-structure"

(Cheeger-Gromov)

Sheeted/non-sheeted decomposition for minimal surfaces

Let (N, g) be a 3-manifold. Consider $n_0 > 0$ a large constant and $\bar{r} > 0$ a small constant. For a minimal surface $\Sigma \subset N$, define at $p \in$ ~~M~~ Σ:

$$s(p) := \sup\{r \leq \bar{r}; \quad \Sigma \cap B(p, r) \text{ is stable}\},$$

and set

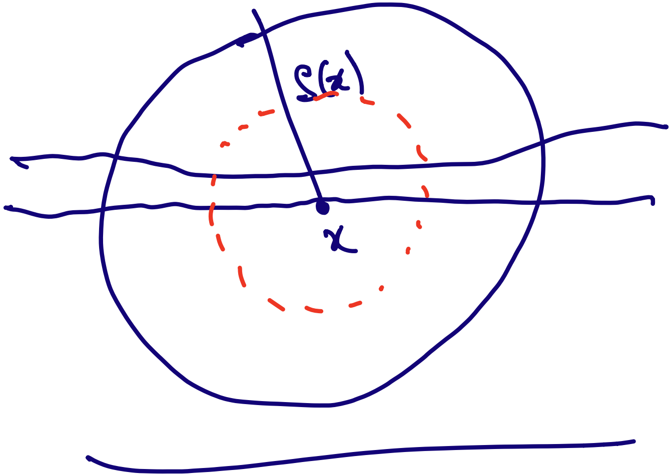
sheeted

$$\Sigma_{>n_0} := \{x \in \Sigma; \quad \Sigma \cap B(x, s(x)) \text{ has area larger than } n_0 s(x)^2\},$$

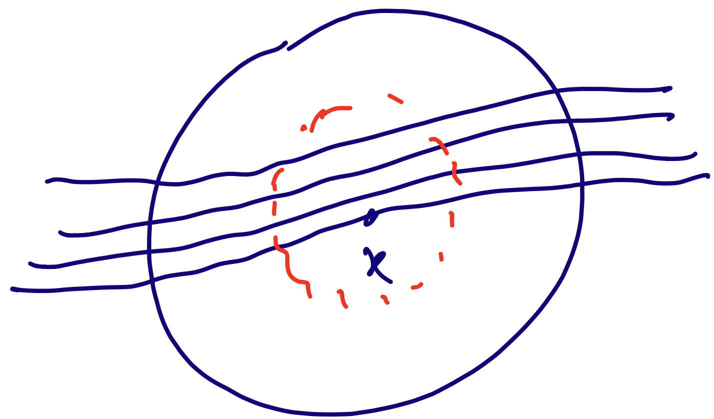
$$\Sigma_{\leq n_0} := \{x \in \Sigma; \quad \Sigma \cap B(x, s(x)) \text{ has area at most } n_0 s(x)^2\}.$$

non-sheeted

here non-sheeted part $\Sigma_{\leq n_0}$ is locally simple



sheeted part $\Sigma_{> n_0}$



Results

Theorem 1: Let (N, g) be a closed 3-manifold, there is a constant $C = C(N, g)$ such that for any minimal surface $\Sigma \subset N$ and $n_0 > 0$,

$$\text{genus}(\Sigma_{\leq n_0}) \leq C n_0 (\text{Morse index}(\Sigma) + 1).$$

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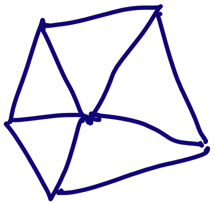
Theorem 2: There are constants C, ϵ, v_0, such that for any closed Einstein 4-manifold (M, g), $M_{>v_0}$ admits a metric h with

$$|\,\text{Rm}_h\,| \leq 1, \quad \text{injrad}_h \geq 1 \text{ and } \text{Vol}(M_{\geq v_0}, h) \leq C\chi(M).$$

Results expressed with triangulation



Theorem 1: Let (N, g) be a closed 3-manifold, there is a constant $C = C(N, g)$ such that for any minimal surface $\Sigma \subset N$ and $n_0 > 0$, $\Sigma_{\leq n_0}$ has a triangulation with degree $\leq C$ and total number of vertices $\leq Cn_0(\text{Morse index}(\Sigma) + 1)$.

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Proof idea for minimal surfaces

Fix $n_0 \gg 1$. To simplify the discussion, we start with a minimal surface $\Sigma \subset N$ which has area $\ll n_0$ (so that $\Sigma_{\ll n_0} = \Sigma$) and Σ is not stable in any ball of radius $\geq \bar{r}$ (depending only on the ambient 3-manifold N).

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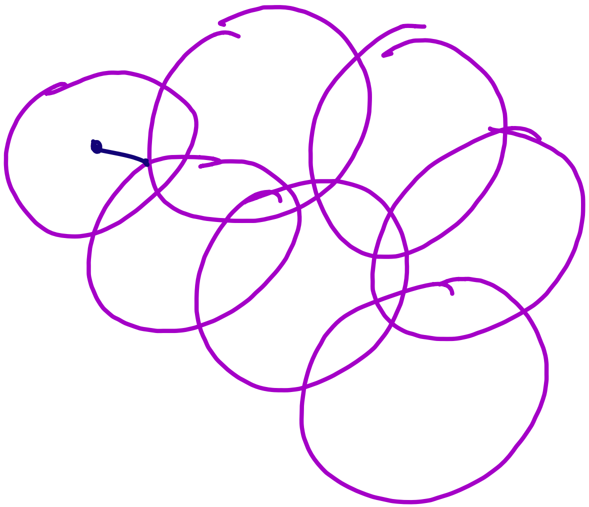

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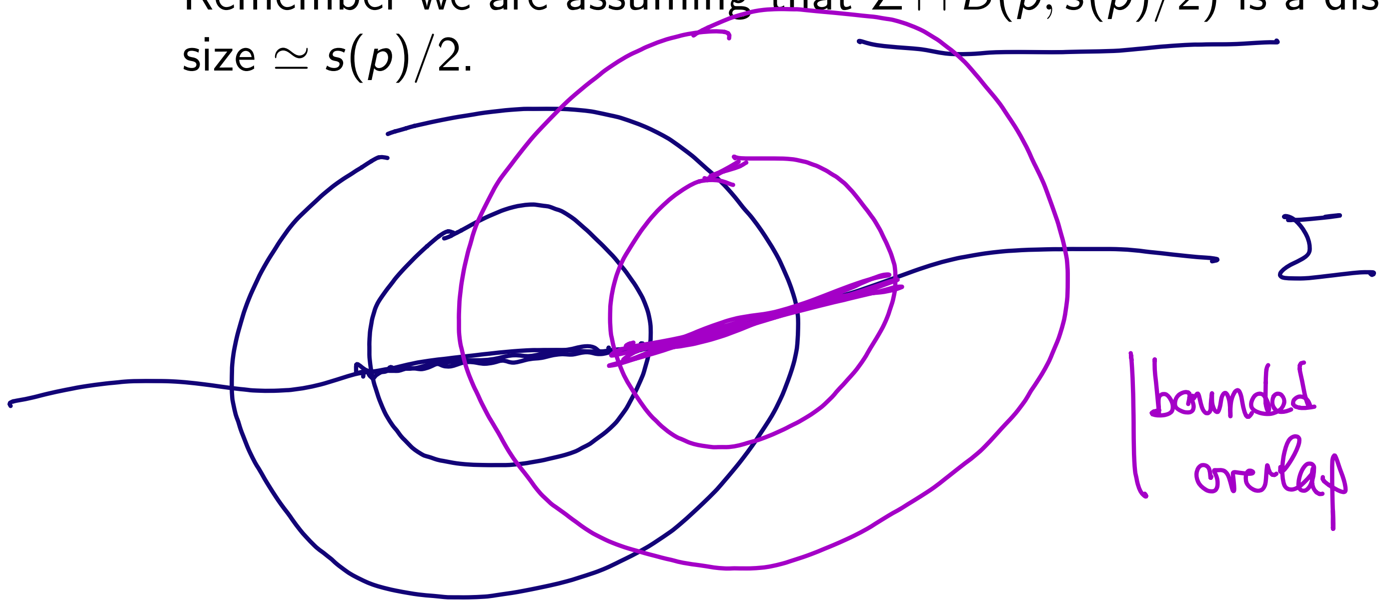
$$s(p) := \sup\{r \leq \bar{r}; \quad \Sigma \cap B(p,r) \text{ is stable}\},$$

so that $\Sigma \cap B(p, s(p))$ is stable but $\Sigma \cap B(p, 2s(p))$ has Morse index ≥ 1.
Remember we are assuming that $\Sigma \cap B(p, s(p)/2)$ is a disk of size $\simeq s(p)/2$.



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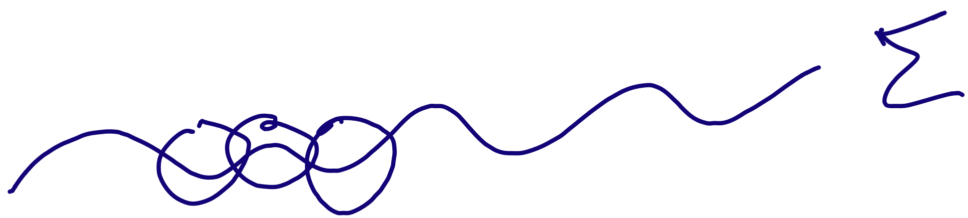
- Cover Σ with q_1 balls of the form $B(p, s(p)/2)$, and q_1 as small as possible. We can construct a triangulation of Σ with $\lesssim q_1$ vertices.



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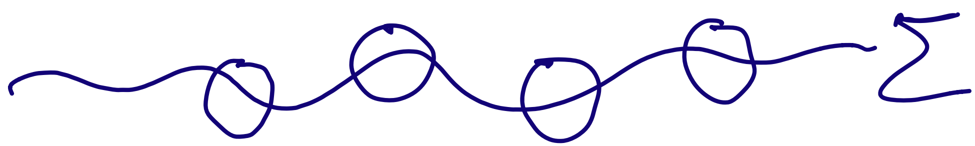
- Find a subfamily of disjoint q_2 balls, where $q_2 \gtrsim q_1$.



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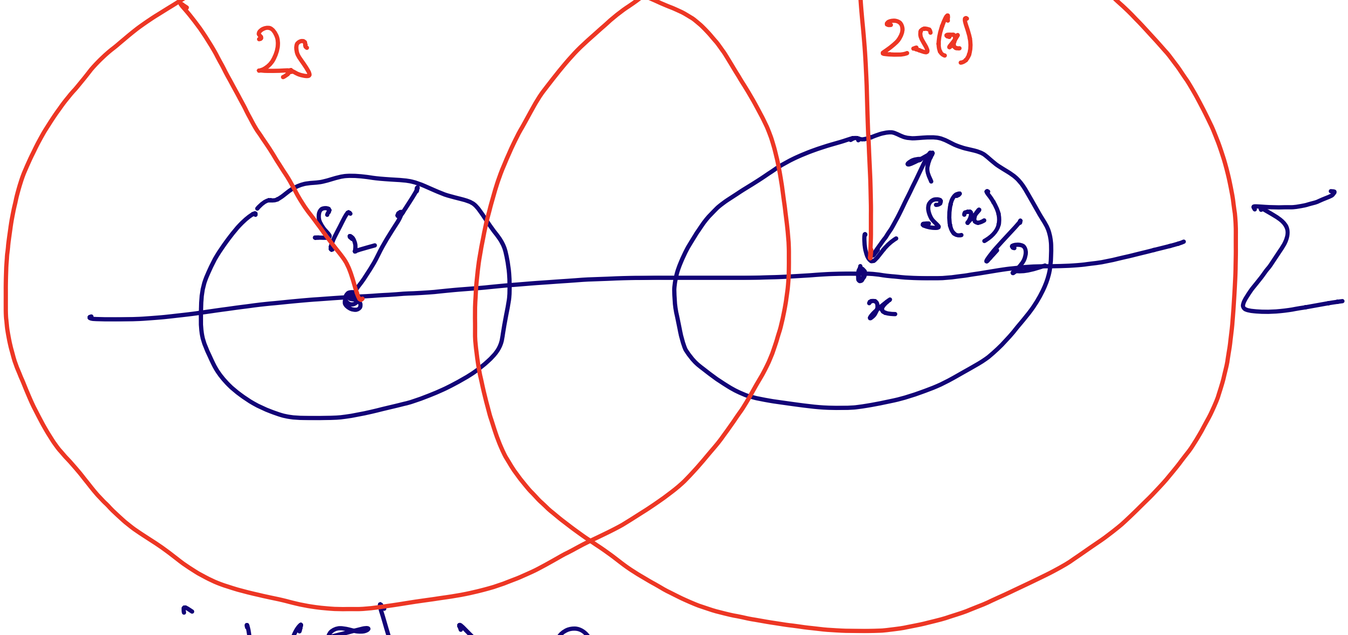
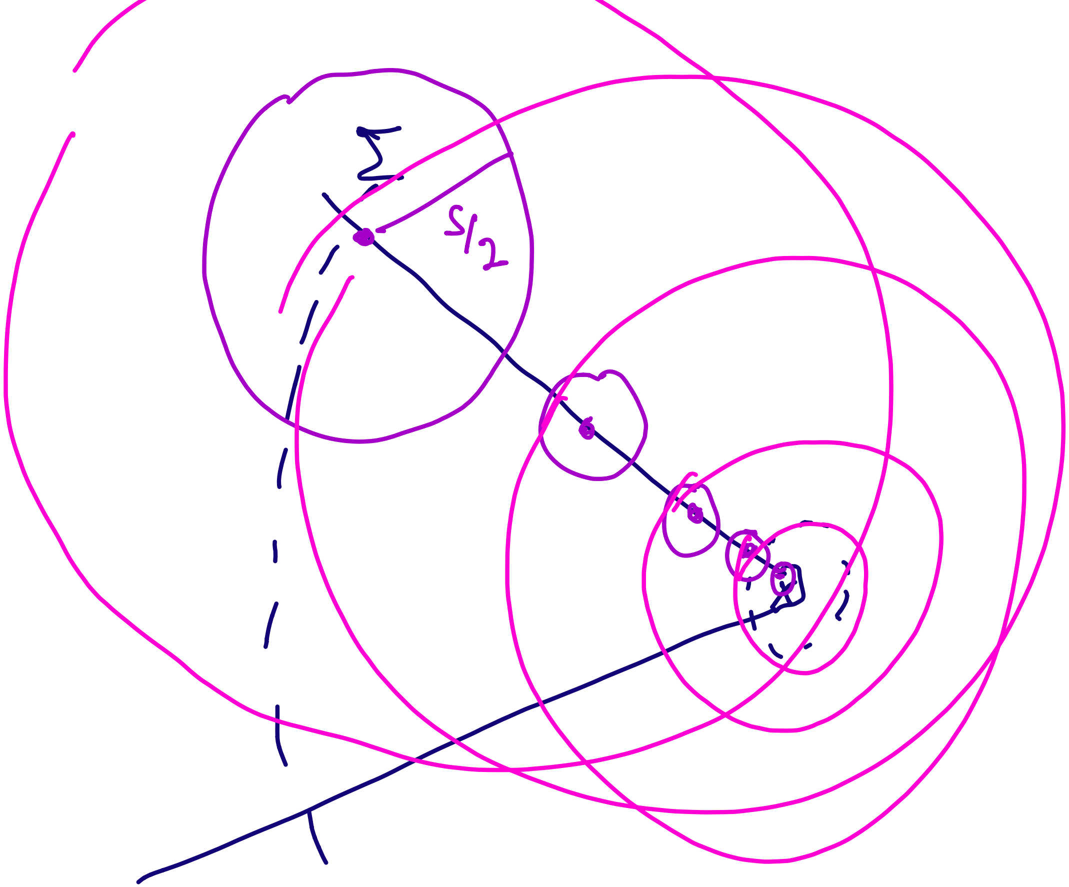
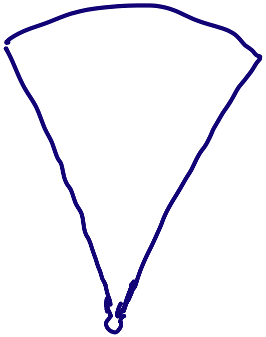
We would be done if we could prove that Morse index$(\Sigma) \gtrsim q_2$.
But the problem is that the balls in the subfamily are not disjoint enough.



$\mathrm{ind}(\Sigma) \geq 2$


Σ
s/2



Proof idea for minimal surfaces

$\Sigma \setminus \{\text{almost conical regions}\}$

In fact, the issue can occur only around almost conical regions. But almost conical regions do not add topology, so we can remove these regions: we cover the complement of these regions by balls $B(x_1, s(x_1)/2), \ldots, B(x_{q_1}, s(x_{q_1})/2)$ then try to get a subfamily of balls $B(y_1, s(y_1)/2), \ldots, B(y_{q_2}, s(y_{q_2})/2)$, so that

$$q_2 \gtrsim q_1 \text{ and } \forall i \neq j, B(y_i, 2s(y_j)) \cap B(y_j, 2s(y_j)) = \varnothing.$$

This is possible by a counting argument.

$\Longrightarrow$ Morse index $\geq q_2 \gtrsim q_1$

Idea of proof for Einstein 4-manifolds

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- Conclude.