On the elastic flow

curves, networks and Willmore flow of tori of revolution

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Statement of the problem

Find curves $f_i : [0, T) \times [0,1] \to \mathbb{R}^n$ and tangential components $\varphi_i : [0, T) \times [0,1] \to \mathbb{R}$ solution to

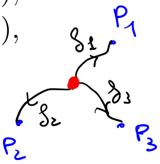
$$\partial_t f_i = -\nabla_s^2 \vec{\kappa}_i - \frac{1}{2}|\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda_i \vec{\kappa}_i + \varphi_i \partial_s f_i \text{ on } (0, T) \times I,$$

for $i = 1, 2, 3$ with boundary conditions

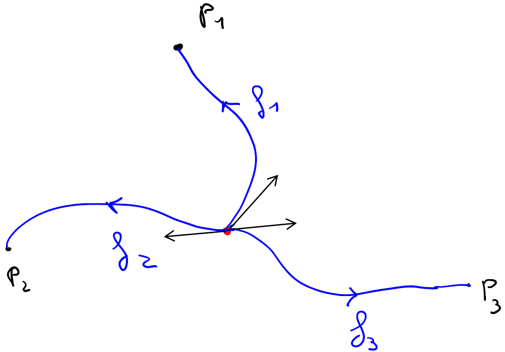
$$\begin{cases} f_i(t,1) = P_i, & \text{for } t \in (0, T), i = 1,2,3, \\ \vec{\kappa}_i(t,1) = 0 = \vec{\kappa}_i(t,0) & \text{for } t \in (0, T), \ i = 1,2,3, \\ f_1(t,0) = f_2(t,0) = f_3(t,0) & \text{for } t \in (0, T), \\ \text{and } \sum_{i=1}^3 (\nabla_s \vec{\kappa}_i(t,0) - \lambda_i \partial_s f_i(t,0)) = 0 & \text{for } t \in (0, T), \end{cases}$$

and initial value

$$f_1(t = 0, \cdot) = f_{i,0} \text{ for } i = 1, 2, 3, \text{ in } [0, 1].$$



Situation



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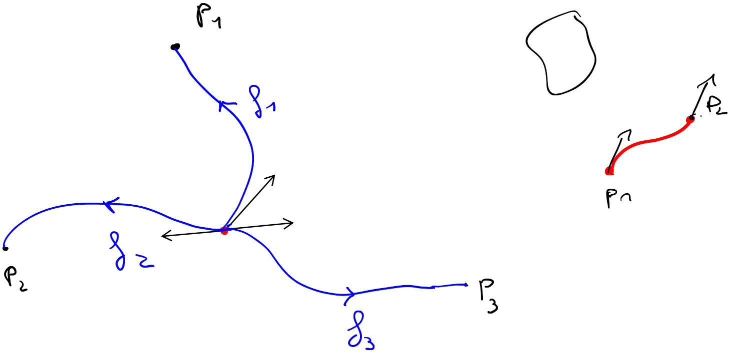

Short time existence under Non-collinearity condition

$$\text{Span}\{\partial_s f_{0,i} : i = 1, 2, 3\} \geqslant 2.$$

Long time existence

Idea: Show that all the norms of the solution remain bounded and then extend the solution. Non-collinearity condition needed at $T_{\max}$.

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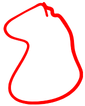
1. Every integration by parts give boundary terms that need to be taken care of.

$$\nabla_t \nabla_s^m k + \nabla_s^{m+4} k = \dots \cdot \nabla_s^m k$$



$$\int \nabla_s^m k \cdot \nabla_s^{m+4} k$$

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   Lemma If $\partial_t f = \vec{V} + \varphi \partial_s f$, $\vec{\phi}$ normal vector field with $\nabla_t \vec{\phi} + \nabla_s^4 \vec{\phi} = Y$, then

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$$\underbrace{\frac{d}{dt}\frac{1}{2}\int_I |\vec{\phi}|^2 ds}_{\text{Gronwall}} + \underbrace{\int_I |\nabla_s^2 \vec{\phi}|^2 ds}_{} =$$

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$$\frac{d}{dt}\frac{1}{2}\int_I |\vec{\phi}|^2 ds + \int_I |\nabla_s^2 \vec{\phi}|^2 ds = -[\langle \vec{\phi}, \nabla_s^3 \vec{\phi}\rangle]_0^1 + [\langle \nabla_s \vec{\phi}, \nabla_s^2 \vec{\phi}\rangle]_0^1 + \int_I \langle Y + \frac{1}{2}\vec{\phi}\varphi_s, \vec{\phi}\rangle ds - \frac{1}{2}\int_I |\vec{\phi}|^2 \langle \vec{\kappa}, \vec{V}\rangle ds,$$

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2. The tangential components φ_i are not known.

$$\partial_t \gamma_i = -\nabla_{L^2} \mathcal{E}_\lambda(\gamma_i) + \varphi_i \partial_s \gamma_i$$

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The junction point

Condition

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$$\partial_t f_i = -\nabla_s^2 \vec{\kappa}_i - \frac{1}{2}|\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda_i \vec{\kappa}_i + \varphi_i \partial_s f_i,$$

and $\vec{\kappa}_i = 0$ at $x = 0$, we get

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$$-\nabla_s^2 \vec{\kappa}_i + \varphi_i \partial_s f_i = -\nabla_s^2 \vec{\kappa}_j + \varphi_j \partial_s f_j\,,$$

that implies

$$\varphi_i(t,0) = -\langle \nabla_s^2 \vec{\kappa}_j, \partial_s f_i \rangle(t,0) + \varphi_j(t,0)\langle \partial_s f_j, \partial_s f_i \rangle(t,0)\,.$$

Condition on the tangential component

$$\varphi_i(t,0) = -\langle \nabla_s^2 \vec{\kappa}_j, \partial_s f_i \rangle(t,0) + \varphi_j(t,0)\langle \partial_s f_j, \partial_s f_i \rangle(t,0)\,.$$

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This can be written as x = 0

$$2\varphi_i - \varphi_{i+1}\langle \partial_s f_{i+1}, \partial_s f_i\rangle - \varphi_{i+2}\langle \partial_s f_{i+2}, \partial_s f_i\rangle = -\langle \nabla_s^2 \vec{\kappa}_{i+1} + \nabla_s^2 \vec{\kappa}_{i+2}, \partial_s f_i\rangle$$

i = 1, 2, 3

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If the non-collinearity condition is satisfied at t, then the system is uniquely solvable and hence the tangential component is determined at $x = 0$.

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we see that $\varphi_i(t,1) = 0$

$$\varphi_i(t,x) = \varphi_i(t,0)\left(1 - \frac{1}{\mathcal{L}(\gamma(t))}\int_0^x |\partial_x \gamma(t,x)|\, dx\right)$$

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What about the interval $(0,1)$?

Since the problem is geometric: we are 'free' to choose the φ_i (for instance by a linear interpolation) on $(0,1)$ using diffeomorphisms of $[0,1]$.

Result

The problem

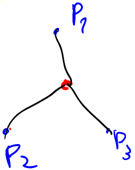
$$\partial_t f_i - \langle \partial_t f_i, \partial_s f_i \rangle \partial_s f_i = -\nabla_s^2 \vec{\kappa}_i - \frac{1}{2}|\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda_i \vec{\kappa}_i \text{ for } i = 1, 2, 3,$$

with boundary conditions

$$\begin{cases} f_i(t,1) = P_i, & , f_1(t,0) = f_2(t,0) = f_3(t,0) \\ \vec{\kappa}_i(t,1) = 0 = \vec{\kappa}_i(t,0) & \sum_{i=1}^{3} (\nabla_s \vec{\kappa}_i(t,0) - \lambda_i \partial_s f_i(t,0)) = 0 \end{cases}$$

and (well chosen) initial value

$$\{f_1(0,\cdot), f_2(0,\cdot), f_3(0,\cdot)\} \text{ in } [0,1]$$



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1. lengths $\mathcal{L}(f_i)$ of the three curves are uniformly bounded from below;
2. and that the dimension of the space spanned by the unit tangents $\partial_s f_i$, $i = 1, 2, 3$, at the triple junctions is bigger or equal to two.

The Willmore flow of tori of revolution

The Willmore energy

Let Σ be a 2-dim. manifold, compact, without boundary and $f : \Sigma \to \mathbb{R}^3$ be a smooth immersion.

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$$W(f) := \frac{1}{4} \int_\Sigma |\vec{H}|^2 \, d\mu$$

$\tilde{H} = \frac{\lambda_1 + \lambda_2}{2}$

with $\vec{H} = H\vec{N}$ the mean curvature vector, $H = \lambda_1 + \lambda_2$ and μ the induced area measure.
Very interesting energy: it is invariant with respect to smooth Möbius transformations.

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References: Thomsen, .., Willmore, Simon, Kuwert-Schätzle, Riviere, Marquez-Neves,....

Hyperbolic half-plane

Pinkall, Bryant-Griffiths, Langer-Singer: Relation between elastica in $\mathbb{H}^2$ and Willmore surfaces of revolution.

Hyperbolic half-plane

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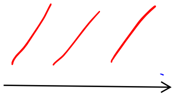
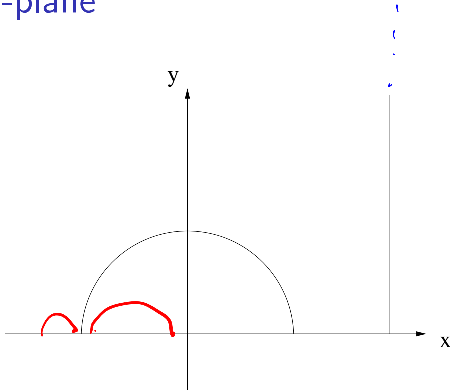
Hyperbolic half-plane $\mathbb{H}^2 = \{(x, z) \in \mathbb{R}^2 \mid z > 0\}$ with metric

$$g_{(x,z)} = \frac{1}{z^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$



$(\mathbb{H}^2, g)$ has constant sectional curvature equal to -1.

Hyperbolic half-plane

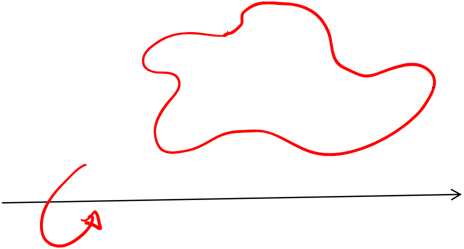


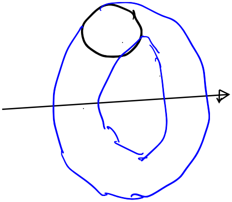
Dilatations are isometry in $\mathbb{H}^2$.
A typical geodesic in $\mathbb{H}^2$ is a half-circle centered at $(p, 0)^t$. There are no closed geodesics.

Surfaces of revolution

Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2_+ := \{(x,z)^t : z > 0\}$ be a closed curve parametrised by arc-length. By rotating the curve around the x-axis we obtain a surface of revolution in $\mathbb{R}^3$

$$h_\gamma : (x, \varphi) \mapsto (\gamma_1(x), \gamma_2(x)\cos(\varphi), \gamma_2(x)\sin(\varphi))^t .$$





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$$h_\gamma : (x,\varphi) \mapsto (\gamma_1(x), \gamma_2(x)\cos(\varphi), \gamma_2(x)\sin(\varphi))^t .$$

$[0,2\pi]$

The induced area element is $\gamma_2(x)\,dxd\varphi$, the principal curvatures are

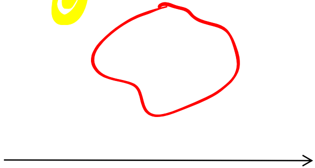
$$\lambda_1 = \gamma_1''(x)\gamma_2'(x) - \gamma_2''(x)\gamma_1'(x) \text{ and } \lambda_2 = \frac{\gamma_1'(x)}{\gamma_2(x)},$$

so that

$$W(h_\gamma) = \frac{1}{4} 2\pi \int_{\mathbb{S}^1} (\lambda_1 + \lambda_2)^2 \gamma_2(x)\,dx.$$

Elastic energy in $\mathbb{H}^2$

The same curve can be considered as a curve in $\mathbb{H}^2$.



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so that

$$\begin{aligned} \mathcal{E}_0(\gamma) &= \int_{\mathbb{S}^1} |\vec{\kappa}|_g^2(x) \frac{1}{\gamma_2(x)} \, \mathrm{d}x \\ &= \int_{\mathbb{S}^1} (\lambda_1 + \lambda_2)^2 \gamma_2(x) \, \mathrm{d}x + \int_{\mathbb{S}^1} 4 \partial_x^2 \gamma_2 \, \mathrm{d}x \\ &= \frac{2}{\pi} W(h_\gamma) \,. \end{aligned}$$

Nice: We have a 1-d flow that decreases the Willmore energy and keeps the rotationally symmetry.

Elastic Flow in $\mathbb{H}^2$

j.w.w. A. Spener.
For $f_0 : \mathbb{S}^1 \to \mathbb{H}^2$ a smooth, regular and closed curve and $\lambda \geqslant 0$, a smooth global solution $f : \mathbb{S}^1 \times [0,\infty) \to \mathbb{H}^2$ to

$$\begin{cases} \partial_t f = -(\nabla^{\perp}_{\partial_s})^2 \vec{\kappa} - \frac{1}{2}|\vec{\kappa}|^2 \vec{\kappa} + \lambda \vec{\kappa} + \vec{\kappa}, & \text{in } \mathbb{S}^1 \times (0,T), \\ f(x,0) = f_0(x) & \text{for } x \in \mathbb{S}^1 . \end{cases}$$

exist.

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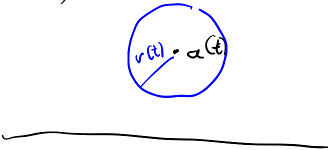
If $\lambda > 0$: control on the length. Really needed?

What about circular solution?

Ansatz:

$$f(x,t) = \begin{pmatrix} 0 \\ a(t) \end{pmatrix} + r(t) \begin{pmatrix} \cos x \\ \sin x \end{pmatrix} .$$



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Lemma For any $\lambda \in \mathbb{R}$ there exists a family of circles that satisfies the elastic flow in $\mathbb{H}^2$. Moreover, for $\lambda > -\frac{1}{2}$, this family converges to the circle which is given by

$$\frac{a}{r} = \sqrt{2(\lambda + 1)}.$$

The case $\lambda = 0$

For $\lambda = 0$, the family of circles converges to the circle

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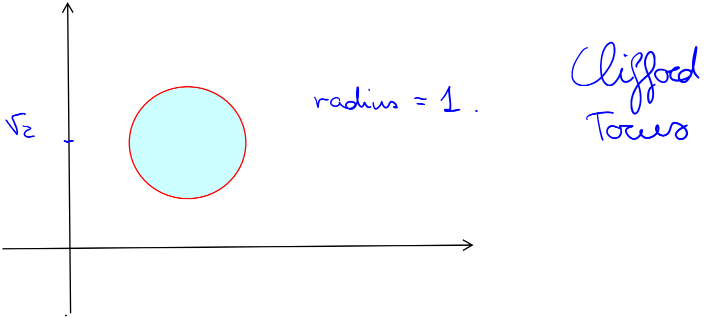

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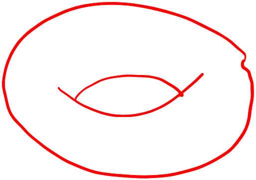
The associated surface of revolution is the Clifford Torus.



A new inequality

Theorem (M. Müller, A. Spener, JGF 2020)
For each $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ s.t.

$$\frac{\mathcal{E}_0(\gamma)}{L_{\mathbb{H}^2}(\gamma)} \geqslant c(\varepsilon)$$

for all immersed closed curves $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$ such that $\mathcal{E}_0(\gamma) \leqslant 16 - \varepsilon$.

$$\mathcal{E}_0(\gamma) = \frac{2}{\pi} \underbrace{\mathcal{W}(h_\gamma)}_{8\pi}$$

$$\| \quad 16$$

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for all immersed closed curves $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$ such that $\mathcal{E}_0(\gamma) \leqslant 16 - \varepsilon$.

Notice that since

$$\mathcal{E}_0(\gamma) = \frac{2}{\pi} W(h_\gamma)\,,$$

the bound 16 for $\mathcal{E}_0$ correspond to 8π for $W(h_\gamma)$.

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Question Can we use our understanding of the elastic energy in $\mathbb{H}^2$ to study the Willmore Flow of Tori of revolution?

Willmore Flow for tori of revolution

Consider $f_0 : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ a torus of revolution, look for a solution $f : (0, T) \times \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ of

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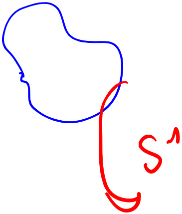

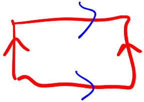

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1. A solution exists in a short interval of time.
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Theorem

j.w.w. M. Müller, R. Schätzle, A. Spener.

Let $f : [0, T) \times \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ be a maximal evolution by Willmore flow such that $f(0)$ is a torus of revolution. Let $(\gamma(t))_{t \in [0,T)}$ be a collection of profile curves of $f(t)$. If

$$\liminf_{t \to T} \mathcal{L}_{\mathbb{H}}(\gamma(t)) < \infty,$$

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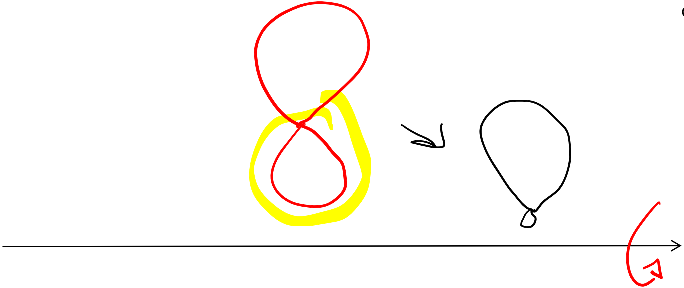
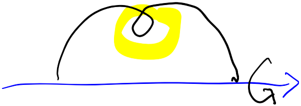
2. There exists a torus of revolution with $W(f_0) > 8\pi$ such that the solution of the Willmore flow either concentrates or the diameter becomes unbounded.

$E_{g_0}(\gamma) \leq 16+\varepsilon$





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A convergence criterium of the Willmore flow

Let Σ be a compact two-dimensional manifold without boundary and let $f : [0, T) \times \Sigma \to \mathbb{R}^3$ be a maximal evolution by the Willmore flow.

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$$r_j := \sup \left\{ r > 0 : \forall x \in \mathbb{R}^3 \text{ one has } \int_{f(t_j)^{-1}(B_r(x))} |A(t_j)|^2 \, \mathrm{d}\mu_{g_{f(t_j)}} \leqslant \epsilon_0 \right\},$$

satisfy $t_j + c_0 r_j^4 < T$ for all $j \in \mathbb{N}$.

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satisfy $t_j + c_0 r_j^4 < T$ for all $j \in \mathbb{N}$. Further, the *concentration rescalings*

$$\tilde{f}_{j,c_0} : \Sigma \to \mathbb{R}^3, \quad \tilde{f}_{j,c_0} := \frac{f(t_j + c_0 r_j^4)}{r_j},$$

satifies that if $(\mathrm{diam}(\tilde{f}_{j,c_0}))_{j\in\mathbb{N}}$ is uniformly bounded, then $T = \infty$ and the Willmore flow converges to a Willmore immersion.

Comparing the conditions

On one hand

$$\lim_{j\to\infty} \mathcal{L}_{\mathbb{H}}(\gamma(t_j)) = \lim_{j\to\infty} \mathcal{L}_{\mathbb{H}}(\tilde{\gamma}(t_j)) < \infty,$$

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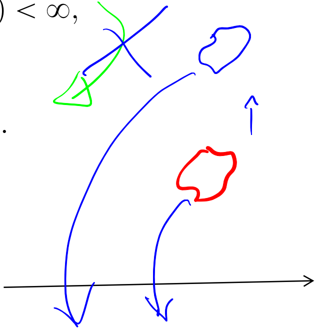
$$(\mathrm{diam}(\tilde{f}_{j,c_0}))_{j\in\mathbb{N}} \leqslant M \quad \forall j.$$

What can go wrong?



Control on the radius of the revolution

Take x_j such that

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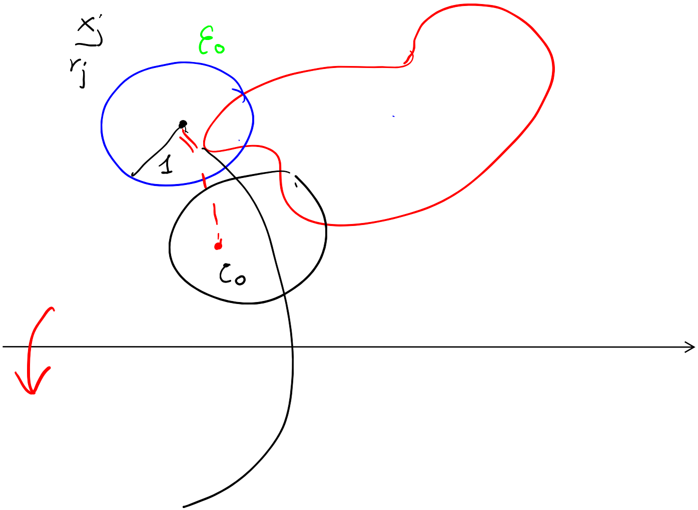
By rescaling

$$\int_{\left(\frac{f(t_j)}{r_j}\right)^{-1}(\overline{B_1(\frac{x_j}{r_j})})} \left| A\left[\frac{f(t_j)}{r_j}\right]\right|^2 \mathrm{d}\mu_{g_{\frac{f(t_j)}{r_j}}} \geqslant \varepsilon_0 .$$

Situation



Situation

Idea If the radius of revolution is too big there is 'space' for too many balls of radius 1.

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With arguments from Müller-Spener

1. Classification of elastica in $\mathbb{H}^2$: there is no closed (free) elastica in $\mathbb{H}^2$ with vanishing total curvature

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3. The total curvature is an invariant along the Willmore flow.
4. Start the Willmore flow from h_{γ_ε}. If the hyperbolic length remains bounded, there is convergence to a Willmore immersion. The limit must be rotationally symmetric and the profile curve a closed elastica with $T[\gamma_\infty] = 0$. ↯

Thank you!