On the elastic flow
curves, networks and Willmore flow of tori of revolution

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The Willmore flow of tori of revolution
- Elastic energy in $\mathbb{H}^2$ and Willmore energy
So far

**Short time existence result**

Let $f_0$ be smooth enough, regular and closed. Then $\exists T > 0$ and a smooth solution $f : (0, T) \times S^1 \rightarrow \mathbb{R}^n$ continuous up to $t = 0$ to

$$\partial_t f = -\left( \nabla_{\frac{1}{\partial s}} \right)^2 \vec{\kappa} - \frac{1}{2} |\vec{\kappa}|^2 \vec{\kappa} + \lambda \vec{\kappa} = -\nabla_{L^2} E_\lambda(f),$$

with $f(0, x) = f_0(x)$ and $E_\lambda(f) = \frac{1}{2} \int_{S^1} |\vec{\kappa}|^2 \, ds + \lambda \int_{S^1} 1 \, ds.$

Let $T_{\text{max}}$ be the maximal existence time of the solution.

**Goal** The solution exists for all times, that is $T_{\text{max}} = \infty$.

**Method** Interpolation inequalities as in Polden and Dziuk-Kuwert-Schätzle. There are no comparison principles available.
So far

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with \( f(0, x) = f_0(x) \) and \( E_\lambda(f) = \frac{1}{2} \int_{S^1} |\vec{\kappa}|^2 \, ds + \lambda \int_{S^1} 1 \, ds. \)

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Step 1: $\| (\nabla_\perp)^m \vec{K} \|_{L^2} \leq c(m, \delta)$, $t \in [\delta, T_{\text{max}})$. 

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$m = 0$: Estimate direct because $L^2$-gradient flow.

$m \geq 1$: By induction.
Assume $T_{\text{max}} < \infty$

**Idea of the proof** Show that in finite time any norm of the solution is bounded. Hence it can be extended.

**Step 1**: $\| (\nabla_{\partial_s}^m \vec{K}) \|_{L^2} \leq c(m, \delta), \; t \in [\delta, T_{\text{max}})$. \( \forall m \in \mathbb{N} \)

$m = 0$: Estimate direct because $L^2$-gradient flow.

\[ E_{\lambda}(g) = \frac{1}{2} \int |\vec{K}|^2 \; ds \quad 2L(g) \leq E_{\lambda}(g_0) \]
\[ \| (\nabla_{\partial s}^\perp)^m \vec{\kappa} \|_{L^2} \leq c(m, \delta), \ t \in [\delta, T_{\text{max}}), \ m \geq 1. \]

Evolution equation for the derivatives of the curvature

\[ \nabla_{\partial t} (\nabla_{\partial s}^\perp)^m \vec{\kappa} + (\nabla_{\partial s}^\perp)^{m+4} \vec{\kappa} = C \langle (\nabla_{\partial s}^\perp)^{m+2} \vec{\kappa}, \vec{\kappa} \rangle \vec{\kappa} + \ldots. \]

Recall that \( \nabla_{\partial s}^\perp \) and \( \nabla_{\partial t} \) do not commute and hence many extra terms appear. One needs a good way to write these terms.

\[ \partial_t \theta = \Theta \left( \nabla_{\partial s}^\perp \right)^2 \vec{e}_t - \frac{1}{2} \| \kappa \|^2 \vec{e}_t + \mathcal{O} \]

\[ \partial_t \phi + (\nabla_{\partial s}^\perp)^2 \vec{e}_t = \ldots. \]
\[ \| (\nabla_{\partial_s} \perp)^m \vec{K} \|_{L^2} \leq c(m, \delta), \ t \in [\delta, T_{\text{max}}), \ m \geq 1. \]

Evolution equation for the derivatives of the curvature

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Recall that \( \nabla_{\partial_s} \perp \) and \( \nabla_{\partial_t} \) do not commute and hence many extra terms appear. One needs a good way to write these terms.

Multiplying by \( (\nabla_{\partial_s} \perp)^m \vec{K} \) and integrating

\[ \frac{1}{2} \int \langle \nabla_{\partial_t} (\nabla_{\partial_s} \perp)^m \vec{K}, (\nabla_{\partial_s} \perp)^m \vec{K} \rangle ds + \int \langle (\nabla_{\partial_s} \perp)^{m+4} \vec{K}, (\nabla_{\partial_s} \perp)^m \vec{K} \rangle ds \]

\[ = C \int \langle (\nabla_{\partial_s} \perp)^{m+2} \vec{K}, \vec{K} \rangle \langle \vec{K}, (\nabla_{\partial_s} \perp)^m \vec{K} \rangle ds + \ldots. \]
\[ \| (\nabla_{\partial_s}^\perp)^m \vec{k} \|_{L^2} \leq c(m, \delta), \ t \in [\delta, T_{\text{max}}), \ m \geq 1. \]

Evolution equation for the derivatives of the curvature

\[ \nabla_{\partial_t}^{\perp} (\nabla_{\partial_s}^{\perp})^m \vec{k} + (\nabla_{\partial_s}^{\perp})^m 4 \vec{k} = C \langle (\nabla_{\partial_s}^{\perp})^{m+2} \vec{k}, \vec{k} \rangle \vec{k} + \ldots \]

Recall that \( \nabla_{\partial_s}^{\perp} \) and \( \nabla_{\partial_t}^{\perp} \) do not commute and hence many extra terms appear. One needs a good way to write these terms.

Multiplying by \( (\nabla_{\partial_s}^{\perp})^m \vec{k} \) and integrating

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\[ = C \int \langle (\nabla_{\partial_s}^{\perp})^{m+2} \vec{k}, \vec{k} \rangle \langle \vec{k}, (\nabla_{\partial_s}^{\perp})^{m} \vec{k} \rangle ds + \ldots \]

Look at the terms appearing.
Estimates...

Integrating by parts

\[ \frac{d}{dt} \frac{1}{2} \int |(\nabla_\partial s)^m \kappa|^2 ds + \int |(\nabla_\partial s)^{m+2} \kappa|^2 ds = C \int \langle (\nabla_\partial s)^{m+2} \kappa, \bar{\kappa} \rangle \langle \bar{\kappa}, (\nabla_\partial s)^{m} \kappa \rangle ds + \text{....} \]

Idea: use interpolation inequalities to control the right-hand side with the left-hand side and use Gromwall's Lemma.
Integrating by parts

\[
\frac{d}{dt} \frac{1}{2} \int |(\nabla_{\partial_s}^\perp)^m \vec{\kappa}|^2 ds + \int |(\nabla_{\partial_s}^\perp)^{m+2} \vec{\kappa}|^2 ds
\]

\[= C \int \langle (\nabla_{\partial_s}^\perp)^{m+2} \vec{\kappa}, \vec{\kappa} \rangle \langle \vec{\kappa}, (\nabla_{\partial_s}^\perp)^m \vec{\kappa} \rangle ds + .... \]

and

\[
\frac{d}{dt} \frac{1}{2} \int |(\nabla_{\partial_s}^\perp)^m \vec{\kappa}|^2 ds + \frac{1}{2} \int |(\nabla_{\partial_s}^\perp)^m \vec{\kappa}|^2 ds + \int |(\nabla_{\partial_s}^\perp)^{m+2} \vec{\kappa}|^2 ds
\]

\[= C \int \langle (\nabla_{\partial_s}^\perp)^{m+2} \vec{\kappa}, \vec{\kappa} \rangle \langle \vec{\kappa}, (\nabla_{\partial_s}^\perp)^m \vec{\kappa} \rangle ds
\]

\[+ \frac{1}{2} \int |(\nabla_{\partial_s}^\perp)^m \vec{\kappa}|^2 ds + .... \]
Estimates...

Integrating by parts

\[
\frac{d}{dt} \frac{1}{2} \int |(\nabla_{\partial s}^\perp)^m \vec{\kappa}|^2 ds + \int |(\nabla_{\partial s}^\perp)^{m+2} \vec{\kappa}|^2 ds
\]

\[
= C \int \langle (\nabla_{\partial s}^\perp)^{m+2} \vec{\kappa}, \vec{\kappa} \rangle \langle \vec{\kappa}, (\nabla_{\partial s}^\perp)^m \vec{\kappa} \rangle ds + \ldots
\]

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\]

\[
= C \int \langle (\nabla_{\partial s}^\perp)^{m+2} \vec{\kappa}, \vec{\kappa} \rangle \langle \vec{\kappa}, (\nabla_{\partial s}^\perp)^m \vec{\kappa} \rangle ds
\]

\[
+ \frac{1}{2} \int |(\nabla_{\partial s}^\perp)^m \vec{\kappa}|^2 ds + \ldots
\]

Idea use interpolation inequalities to control the r.h.s. with the l.h.s. and use Gromwall’s Lemma

Anna Dall’Acqua  On the elastic flow
Interpolation inequalities

One finds constants $c = c(i, k, p, \frac{1}{L})$ such that

$$
\|(\nabla \frac{1}{\partial s})^i \vec{\kappa}\|_{L^p} \leq c \|(\nabla \frac{1}{\partial s})^k \vec{\kappa}\|^\alpha_{L^2} \|\vec{\kappa}\|_{L^2}^{1-\alpha} + c \|\vec{\kappa}\|_{g\|L^2}
$$

for $p \in [2, \infty]$, $i, k \in \mathbb{N}$, $i < k$, $L$ the length of the curve and

$$
\alpha = \frac{i + 1/2 - 1/p}{k}.
$$
Interpolation inequalities

One finds constants $c = c(i, k, p, \frac{1}{L})$ such that

$$
\| (\nabla_{\partial_s}^l)^i \vec{\kappa} \|_{L^p} \leq c \| (\nabla_{\partial_s}^k)^k \vec{\kappa} \|_{L^2}^{\alpha} \| \vec{\kappa} \|_{L^2}^{1-\alpha} + c \| |\vec{\kappa}|_g \|_{L^2}
$$

for $p \in [2, \infty]$, $i, k \in \mathbb{N}$, $i < k$, $L$ the length of the curve and

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$$

Need a bound from below on the length.
Interpolation inequalities

One finds constants $c = c(i, k, p, \frac{1}{L})$ such that

$$
\left\| \left( \nabla \frac{1}{\partial s} \right)^i \vec{K} \right\|_{L^p} \leq c \left\| \left( \nabla \frac{1}{\partial s} \right)^k \vec{K} \right\|_{L^2} \| \vec{K} \|^\alpha_{L^2} + c \| \vec{K} \|_{g} \|_{L^2}
$$

for $p \in [2, \infty]$, $i, k \in \mathbb{N}$, $i < k$, $L$ the length of the curve and

$$
\alpha = \frac{i + 1/2 - 1/p}{k}.
$$

Need a bound from below on the length. Theorem of Fenchel\(^1\):

$$
(2\pi)^2 \leq \left( \int |\vec{K}| \, ds \right)^2 \leq \mathcal{E}(f(t))L(f(t)) \leq \mathcal{E}_\lambda(f_0)L(f(t)).
$$

\(^1\)1905-1988, from 1933 worked (mostly) at the University of Copenhagen.
From

\[
\frac{d}{dt} \frac{1}{2} \int |(\nabla_{\partial_s}^\perp)^m \kappa|^2 ds + \frac{1}{2} \int |(\nabla_{\partial_s}^\perp)^m \kappa|^2 ds + \int |(\nabla_{\partial_s}^\perp)^{m+2} \kappa|^2 ds \\
= C \int \langle (\nabla_{\partial_s}^\perp)^{m+2} \kappa, \kappa \rangle \langle \kappa, (\nabla_{\partial_s}^\perp)^m \kappa \rangle ds \\
+ \frac{1}{2} \int |(\nabla_{\partial_s}^\perp)^m \kappa|^2 ds + \ldots.
\]

one gets with interpolation inequalities and the induction assumption to

\[
\frac{d}{dt} \frac{1}{2} \int |(\nabla_{\partial_s}^\perp)^m \kappa|^2 ds + \frac{1}{2} \int |(\nabla_{\partial_s}^\perp)^m \kappa|^2 ds \leq C.
\]

A Gromwall-type argument yields the $L^2$-bound of the derivatives.
From geometric to analytic estimates

So far \( \| (\nabla_{\partial_s} \|^m \kappa \|_{L^2} \leq c(m, \delta), \ t \in [\delta, T_{\text{max}}), \ m \in \mathbb{N}. \)

\(^2\)Similar principle for higher order derivatives
From geometric to analytic estimates

So far \( \|(\nabla_{\partial_s}^\perp)^m \vec{\kappa}\|_2 \leq c(m, \delta), \ t \in [\delta, T_{\text{max}}), \ m \in \mathbb{N}. \)

First issue: get rid of the projection: use that

\[
\partial_s \vec{\kappa} = \nabla_{\partial_s}^\perp \vec{\kappa} - |\vec{\kappa}|^2 \partial_s f.
\]

and similarly \( \partial_s^m \vec{\kappa} = (\nabla_{\partial_s}^\perp)^m \vec{\kappa} + \text{l.o.t.} \).

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\[
\| \partial_s^m \vec{\kappa} \|_{L^2} \leq c(m, \delta), \ t \in [\delta, T_{\text{max}}), \ m \in \mathbb{N}.
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\(^2\)Similar principle for higher order derivatives
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First issue: get rid of the projection: use that

\[
\partial_s \vec{\kappa} = \nabla_{\partial_s}^\perp \vec{\kappa} - |\vec{\kappa}|^2 \partial_s f.
\]

and similarly \( \partial^m_s \vec{\kappa} = (\nabla_{\partial_s}^\perp)^m \vec{\kappa} + \text{l.o.t.} \). Then,

\[
\| \partial^m_s \vec{\kappa} \|_{L^2} \leq c(m, \delta), \ t \in [\delta, T_{\text{max}}), \ m \in \mathbb{N}.
\]

Second issue: get rid of \( \partial_s \) Follows from

\[
\partial_t |\partial_x f| = -\langle \vec{\kappa}, \partial_t f \rangle |\partial_x f|,
\]

and the regularity of the initial datum\(^2\).

\(^2\)Similar principle for higher order derivatives
Control from above of the length: in finite time
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For $\lambda > 0$: an estimate follows from the gradient flow property

$$\lambda L(f(t)) \leq \mathcal{E}_\lambda(f(t)) \leq \mathcal{E}_\lambda(f(0)).$$
Control from above of the length: in finite time

For $\lambda > 0$: an estimate follows from the gradient flow property

$$
\lambda L(f(t)) \leq E_\lambda(f(t)) \leq E_\lambda(f(0)).
$$

For $\lambda = 0$:

$$
\frac{d}{dt} \mathcal{L}[f] = \int_{S^1} \langle \vec{\kappa}, \partial_t f \rangle \leq C.
$$

$$
\mathcal{L}(g(t)) = \mathcal{L}(g(0)) + \int_0^t \mathcal{L}(g(s)) \, ds
$$
Control from above of the length: in finite time

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$$\lambda L(f(t)) \leq \mathcal{E}_\lambda(f(t)) \leq \mathcal{E}_\lambda(f(0)).$$

For $\lambda = 0$:

$$\frac{d}{dt} \mathcal{L}[f] = \int_{S^1} \langle \mathbf{k}, \partial_t f \rangle \leq C.$$

Together with control of $\|f\|_\infty$ by

$$f(t, x) = f(0, x) + \int_0^t \partial_t f(s, x) \, ds,$$

we get uniform bounds in $W^{m,2} \subset C^{m-1,\alpha}$.
Subconvergence

Lemma

For $\lambda > 0$, as $t_i \to \infty$ there exist $p_i \in \mathbb{R}^n$ such the curves $f(\cdot, t_{ik}) - p_{ik}$ converge, when reparametrized by constant speed, to a critical point of $E_\lambda$. 
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Let $(f(\cdot, t_i))_i$ be parametrised by constant speed on $[0, 1]$. Since $\lambda > 0$, the length of $(f(\cdot, t_i))_i$ are uniformly bounded by $L < \infty$. 
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Let $(f(\cdot, t_i))_i$ be parametrised by constant speed on $[0, 1]$. Since $\lambda > 0$, the length of $(f(\cdot, t_i))_i$ are uniformly bounded by $L < \infty$. Let $(p_i)_i \in \mathbb{R}^n$ s.t.

$$\hat{f}(\cdot, t_i) = f(\cdot, t_i) - p_i$$

go through the point $0 \in \mathbb{R}^n$. 
By construction $(\hat{f}(t_i))_i$ are uniformly bounded in $C^m$ and stay in a compact subset of $\mathbb{R}^n$. Hence there exists a subsequence $(t_{i_j})_j$ and $\hat{f}$ smooth such that $\hat{f}(t_{i_j}) \to \hat{f}$. 

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On the elastic flow
By construction \((\hat{f}(t_i))_i\) are uniformly bounded in \(C^m\) and stay in a compact subset of \(\mathbb{R}^n\). Hence there exists a subsequence \((t_{ij})_j\) and \(\hat{f}\) smooth such that \(\hat{f}(t_{ij}) \to \hat{f}\).

It remains to show that \(\hat{f}\) is a critical point.
By construction \((\hat{f}(t_i))_i\) are uniformly bounded in \(C^m\) and stay in a compact subset of \(\mathbb{R}^n\). Hence there exists a subsequence \((t_{ij})_j\) and \(\hat{f}\) smooth such that \(\hat{f}(t_{ij}) \to \hat{f}\).

It remains to show that \(\hat{f}\) is a critical point. Since

\[
  u(t) = \int_I |\partial_t f|^2 \, ds = -\frac{d}{dt} \mathcal{E}_\lambda(f),
\]

we have that \(u \in L^1(0, \infty)\).
By construction \((\hat{f}(t_i))_i\) are uniformly bounded in \(C^m\) and stay in a compact subset of \(\mathbb{R}^n\). Hence there exists a subsequence \((t_{ij})_j\) and \(\hat{f}\) smooth such that \(\hat{f}(t_{ij}) \to \hat{f}\).

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\[
 u(t) = \int_I |\partial_t f|^2 ds = -\frac{d}{dt} \mathcal{E}_\lambda(f),
\]

we have that \(u \in L^1(0, \infty)\). Moreover, \(|u'(t)|\) is uniformly bounded. It follows that

\[
 \lim_{t \to \infty} u(t) = 0.
\]

Thus \(\hat{f}\) is a critical point for \(\mathcal{E}_\lambda\).
Goal: Understand the elastic flow of networks.
The easiest situation: star-shaped network

Consider three smooth regular curves $f_i: I \rightarrow \mathbb{R}^n$, $i = 1, 2, 3$, such that

1. The end-points are fixed: $f_1(1) = P_1$, $f_2(1) = P_2$, $f_3(1) = P_3$, with given distinct points $P_i$, $i = 1, 2, 3$, in $\mathbb{R}^n$.
2. The curves start at the same point $f_1(0) = f_2(0) = f_3(0)$.

For $\lambda > 0$, the elastic energy of the network is $E_\lambda(\Gamma) = \sum_{i=1}^{3} E_\lambda(f_i)$. 

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On the elastic flow
The easiest situation: star-shaped network

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2. The curves start at the same point

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For $\lambda > 0$, the elastic energy of the network is

\[ E_\lambda(\Gamma) = \sum_{i=1}^{3} E_\lambda(f_i). \]
First idea

How to model the flow of a network?
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How to model the flow of a network?
Let \( f_{0,i} \) be smooth enough, regular such that \( \{ f_{0,1}, f_{0,2}, f_{0,3} \} \) is a star-shaped network, and consider

\[
\begin{align*}
\partial_t f_i &= - \left( \nabla_{\frac{1}{\partial_s}} \right)^2 \vec{\kappa}_i - \frac{1}{2} |\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda \vec{\kappa}_i, \\
n_{i}(t, 0) &= n_j(t, 0) \\
n_i(t, 1) &= P_i
\end{align*}
\]

and \( f(0, x) = f_0(x) \).
First idea

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\[
\begin{aligned}
\partial_t f_i &= - (\nabla \frac{1}{\partial s})^2 \kappa_i - \frac{1}{2} |\kappa_i|^2 \kappa_i + \lambda \kappa_i, \\
&\quad \text{on } (0, T) \times [0, 1], \\
f_i(t, 0) &= f_j(t, 0) \quad \text{on } (0, T), i, j = 1, 2, 3, \\
f_i(t, 1) &= P_i \quad \text{on } (0, T), i = 1, 2, 3.
\end{aligned}
\]

and \( f(0, x) = f_0(x) \).

Some problems
First idea

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Let \( f_{0,i} \) be smooth enough, regular such that \( \{ f_{0,1}, f_{0,2}, f_{0,3} \} \) is a star-shaped network, and consider

\[
\begin{align*}
\partial_t f_i &= -\left( \nabla \frac{1}{\partial_s} \right)^2 \vec{\kappa}_i - \frac{1}{2} |\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda \vec{\kappa}_i, & \text{on } (0, T) \times [0, 1], \\
f_i(t, 0) &= f_j(t, 0) & \text{on } (0, T), i, j = 1, 2, 3, \\
f_i(t, 1) &= P_i & \text{on } (0, T), i = 1, 2, 3.
\end{align*}
\]

and \( f(0, x) = f_0(x) \).

Some problems

1. Number of boundary conditions
First idea

How to model the flow of a network? Let $f_{0,i}$ be smooth enough, regular such that $\{f_{0,1}, f_{0,2}, f_{0,3}\}$ is a star-shaped network, and consider

$$
\begin{align*}
\partial_t f_i &= -(\nabla_\perp \frac{1}{\partial s})^2 \vec{\kappa}_i - \frac{1}{2} |\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda \vec{\kappa}_i, \\
f_i(t, 0) &= f_j(t, 0), \\
f_i(t, 1) &= P_i
\end{align*}
$$

on $(0, T) \times [0, 1]$, on $(0, T), i, j = 1, 2, 3$, on $(0, T), i = 1, 2, 3$.

and $f(0, x) = f_0(x)$.

Some problems

1. Number of boundary conditions
2. Does the solution (if it exists) describe a network? That is, what about the Topology?
Elastic flow of curves
Long time existence
Elastic flow of networks
The Willmore flow of tori of revolution

What is a network
Setting
Existence of a solution
Convergence

Situation

$g_i: [0,1] \rightarrow \mathbb{R}^n$

$\Theta_t g_i = -\nabla_u E(g_i) + \frac{\partial}{\partial u}$

Anna Dall’Acqua
On the elastic flow
Boundary conditions: a toy computation

Consider

\[ E(u, v) = \frac{1}{2} \int_0^1 (u_{xx})^2 + (v_{xx})^2 \, dx \]

defined for \( u, v \in W^{2,2}(0, 1) \) such that

\[ u(0) = v(0) \text{ and } u(1) = 0, \ v(1) = 1. \]
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First variation:

\[ \left. \frac{d}{dt} E(u + t\varphi, v + t\psi) \right|_{t=0} = \frac{1}{2} \left. \frac{d}{dt} \int_{0}^{1} (u_{xx} + t\varphi_{xx})^2 + (v_{xx} + t\psi_{xx})^2 \, dx \right|_{t=0} \]
Boundary conditions: a toy computation...

For \( u, v \in W^{2,2}(0, 1) \) with \( u(0) = v(0), \ u(1) = 0, \ v(1) = 1, \)

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\[= \int_0^1 u_{xx} \varphi_{xx} + v_{xx} \psi_{xx} \, dx \]
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First variation:

$$\left. \frac{d}{dt} E(u + t\varphi, v + t\psi) \right|_{t=0} = \int_0^1 u_{xx}\varphi_{xx} + v_{xx}\psi_{xx} \, dx$$

$$= \left. u_{xx}\varphi_x \right|_0^1 - \left. u_{xxx}\varphi \right|_0^1 + \left. v_{xx}\psi_x \right|_0^1 - \left. v_{xxx}\psi \right|_0^1$$

$$+ \int_0^1 u_{xxxx}\varphi + v_{xxxx}\psi \, dx$$
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for $(\varphi, \psi)$ admissible variation.
Boundary conditions: a toy computation......

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Boundary conditions: a toy computation

For \( u, \nu \in W^{2,2}(0,1) \) with \( u(0) = \nu(0), \ u(1) = 0, \nu(1) = 1, \)

\[
E(u, \nu) = \frac{1}{2} \int_{0}^{1} (u_{xx})^2 + (\nu_{xx})^2 \, dx
\]

\[
\frac{d}{dt} E(u + t\varphi, \nu + t\psi) \bigg|_{t=0} = u_{xx}\varphi_x \bigg|_{0}^{1} - u_{xxx}\varphi \bigg|_{0}^{1} + \nu_{xx}\psi_x \bigg|_{0}^{1} - \nu_{xxx}\psi \bigg|_{0}^{1} + \int_{0}^{1} u_{xxxx}\varphi + \nu_{xxxx}\psi \, dx \equiv 0
\]

for \((\varphi, \psi)\) admissible variation. That is,

\[
\varphi(1) = 0 = \psi(1) \text{ und } \varphi(0) = \psi(0).
\]

Natural boundary conditions:

\[
u_{xx}(0) = u_{xx}(1) = 0 = \nu_{xx}(0) = u_{xx}(1) \text{ und } u_{xxx}(0) = \nu_{xxx}(0).
\]
Statement of the problem

Find curves $f_i : [0, T) \times [0, 1] \to \mathbb{R}^n$ solution to

$$\partial_t f_i = -\nabla^2_s \vec{\kappa}_i - \frac{1}{2} |\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda_i \vec{\kappa}_i + \varphi_i \partial_s f_i \text{ on } (0, T) \times I,$$

for $i = 1, 2, 3$ with boundary conditions

\[
\begin{cases}
  f_i(t, 1) = P_i, \\
  \vec{\kappa}_i(t, 1) = 0 = \vec{\kappa}_i(t, 0) \\
  f_1(t, 0) = f_2(t, 0) = f_3(t, 0) \\
  \text{and } \sum_{i=1}^{3} (\nabla_s \vec{\kappa}_i(t, 0) - \lambda_i \partial_s f_i(t, 0)) = 0
\end{cases}
\]

for $t \in (0, T)$, $i = 1, 2, 3$, for $t \in (0, T)$,

and initial value

$$f_1(t = 0, \cdot) = f_{i,0} \text{ for } i = 1, 2, 3, \text{ in } [0, 1].$$
Better statement of the problem

Find curves \( f_i : [0, T) \times [0, 1] \to \mathbb{R}^n \) and tangential components \( \varphi_i : [0, T) \times [0, 1] \to \mathbb{R} \) solution to

\[
\partial_t f_i = -\nabla^2_s \tilde{k}_i - \frac{1}{2}|\tilde{k}_i|^2 \tilde{k}_i + \lambda_i \tilde{k}_i + \varphi_i \partial_s f_i \quad \text{on} \quad (0, T) \times I,
\]

for \( i = 1, 2, 3 \) with boundary condition as before and initial value as before.
Better statement of the problem

Find curves $f_i : [0, T) \times [0, 1] \to \mathbb{R}^n$ and tangential components $\varphi_i : [0, T) \times [0, 1] \to \mathbb{R}$ solution to

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Now the tangential component is part of the problem!
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on $(0, T) \times I$, for $i = 1, 2, 3$ with boundary condition as before and initial value as before.

Now the tangential component is part of the problem!
Joint work with C.-C. Lin and P. Pozzi.
Also: H. Garcke, J. Mendez and A. Pluda.
Short time existence
Short time existence

1. Choose a tangential component to get parabolicity.
Short time existence

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2. Linearize the problem: also the boundary conditions.
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3. Well-posedness of the linear problem: **Non-collinearity condition**
   \[
   \text{Span}\{\partial_s f_0,i : i = 1, 2, 3\} \geq 2.
   \]
Short time existence

1. Choose a tangential component to get parabolicity.
2. Linearize the problem: also the boundary conditions.
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   \[ \text{Span}\{\partial_s f_0, i : i = 1, 2, 3\} \geq 2. \]
Long time existence

**Idea:** Show that all the norms of the solution remain bounded and then extend the solution. Non-collinearity condition needed at $T_{\text{max}}$. 

Lemma If $\dot{\vec{f}} = \vec{V} + \phi \partial_s f$, $\vec{\phi}$ normal vector field with $\nabla_{\vec{t}} \vec{\phi} + \nabla_{\vec{s}} \vec{\phi} = Y$, then

$$\frac{d}{dt} \frac{1}{2} \int_I |\vec{\phi}|^2 \, ds + \int_I |\nabla_{\vec{s}} \vec{\phi}|^2 \, ds = -\left[ \langle \vec{\phi}, \nabla_{\vec{s}}^3 \vec{\phi} \rangle \right]_{10} + \left[ \langle \nabla_{\vec{s}} \vec{\phi}, \nabla_{\vec{s}}^2 \vec{\phi} \rangle \right]_{10} + \int_I \langle Y + \frac{1}{2} \vec{\phi} \phi, \vec{\phi} \rangle \, ds - \frac{1}{2} \int_I |\vec{\phi}|^2 \langle \vec{\kappa}, \vec{V} \rangle \, ds,$$