On the elastic flow

curves, networks and Willmore flow of tori of revolution

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So far

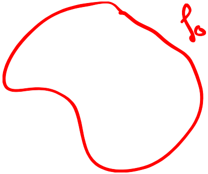
Short time existence result

Let f_0 be smooth enough, regular and closed. Then $\exists T > 0$ and a smooth solution $f : (0, T) \times \mathbb{S}^1 \to \mathbb{R}^n$ continuous up to $t = 0$ to

$$\partial_t f = -(\nabla^{\perp}_{\partial_s})^2 \vec{\kappa} - \frac{1}{2}|\vec{\kappa}|^2 \vec{\kappa} + \lambda \vec{\kappa} = -\nabla_{L^2} \mathcal{E}_{\lambda}(f),$$

with $f(0, x) = f_0(x)$ and $\mathcal{E}_{\lambda}(f) = \frac{1}{2} \int_{\mathbb{S}^1} |\vec{\kappa}|^2 \, ds + \lambda \int_{\mathbb{S}^1} 1 \, ds.$

$\lambda \geq 0$



$f_0 : \mathbb{S}^1 \to \mathbb{R}^n$

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Method Interpolation inequalities as in Polden and Dziuk-Kuwert-Schätzle. There are no comparison principles available.

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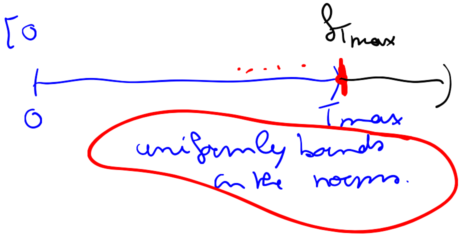
Idea of the proof Show that in finite time any norm of the solution is bounded. Hence it can be extended.



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$m = 0$: Estimate direct because L^2-gradient flow.

$m \geq 1$: By induction.

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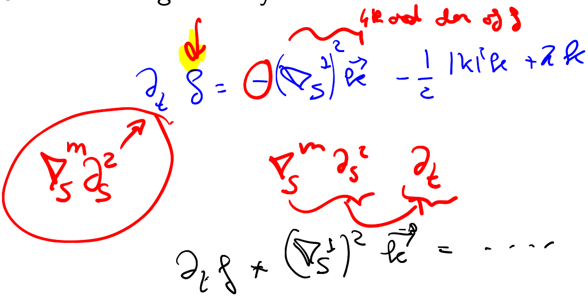
$\mathcal{E}_\lambda(\gamma) = \frac{1}{2}\int |\kappa|^2 \, ds + \lambda \mathcal{L}(\gamma) \le \mathcal{E}_\lambda(\gamma_0)$

$$\|(\nabla^{\perp}_{\partial_s})^m \vec{\kappa}\|_{L^2} \leq c(m,\delta),\ t \in [\delta, T_{\max}),\ m \geq 1.$$

Evolution equation for the derivatives of the curvature

$$\nabla^{\perp}_{\partial_t}(\nabla^{\perp}_{\partial_s})^m \vec{\kappa} + (\nabla^{\perp}_{\partial_s})^{m+4}\vec{\kappa} = C\langle(\nabla^{\perp}_{\partial_s})^{m+2}\vec{\kappa}, \vec{\kappa}\rangle\vec{\kappa} + \ldots.$$

Recall that $\nabla^{\perp}_{\partial_s}$ and $\nabla^{\perp}_{\partial_t}$ do not commute and hence many extra terms appear. One needs a good way to write these terms.



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$$\frac{1}{2}\int \langle \nabla^{\perp}_{\partial_t}(\nabla^{\perp}_{\partial_s})^m \vec{\kappa}, (\nabla^{\perp}_{\partial_s})^m\vec{\kappa}\rangle ds + \int \langle (\nabla^{\perp}_{\partial_s})^{m+4}\vec{\kappa}, (\nabla^{\perp}_{\partial_s})^m\vec{\kappa}\rangle ds$$
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$$\|(\nabla_{\partial_s}^{\perp})^m \vec{\kappa}\|_{L^2} \leq c(m,\delta),\ t \in [\delta, T_{\max}),\ m \geq 1.$$

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Look at the terms appearing.

Estimates...

Integrating by parts

$$\frac{d}{dt}\frac{1}{2}\int |(\nabla^{\perp}_{\partial_s})^m \vec{\kappa}|^2 ds + \int |(\nabla^{\perp}_{\partial_s})^{m+2} \vec{\kappa}|^2 ds$$
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Idea use interpolation inequalities to control the r.h.s. with the l.h.s. and use Gromwall's Lemma

Interpolation inequalities

One finds constants $c = c(i, k, p, \frac{1}{L})$ such that

$$\|(\nabla_{\partial_s}^{\perp})^{i}\vec{\kappa}\|_{L^p} \leq c\|(\nabla_{\partial_s}^{\perp})^{k}\vec{\kappa}\|_{L^2}^{\alpha}\|\vec{\kappa}\|_{L^2}^{1-\alpha} + c\||\vec{\kappa}|_g\|_{L^2}$$

for $p \in [2, \infty]$, $i, k \in \mathbb{N}$, $i < k$, L the length of the curve and

$$\alpha = \frac{i + 1/2 - 1/p}{k}.$$

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Need a bound from below on the length. Theorem of Fenchel[1]:

$$(2\pi)^2 \leq \left(\int |\vec{\kappa}| \, ds \right)^2 \leq \mathcal{E}(f(t)) L(f(t)) \leq \mathcal{E}_{\lambda}(f_0) L(f(t)).$$

[1] 1905-1988, from 1933 worked (mostly) at the University of Copenhagen.

From

$$\frac{d}{dt}\frac{1}{2}\int |(\nabla_{\partial_s}^{\perp})^m \vec{\kappa}|^2 ds + \frac{1}{2}\int |(\nabla_{\partial_s}^{\perp})^m \vec{\kappa}|^2 ds + \int |(\nabla_{\partial_s}^{\perp})^{m+2} \vec{\kappa}|^2 ds$$
$$= C\int \langle (\nabla_{\partial_s}^{\perp})^{m+2}\vec{\kappa}, \vec{\kappa}\rangle \langle \vec{\kappa}, (\nabla_{\partial_s}^{\perp})^m \vec{\kappa}\rangle ds$$
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one gets with interpolation inequalities and the induction assumption to

$$\frac{d}{dt}\frac{1}{2}\int |(\nabla_{\partial_s}^{\perp})^m \vec{\kappa}|^2 ds + \frac{1}{2}\int |(\nabla_{\partial_s}^{\perp})^m \vec{\kappa}|^2 ds \leq C\,.$$

A Gromwall-type argument yields the L^2-bound of the derivatives.

From geometric to analytic estimates

So far $\|(\nabla^{\perp}_{\partial_s})^m \vec{\kappa}\|_{L^2} \leq c(m,\delta)$, $t \in [\delta, T_{\max})$, $m \in \mathbb{N}$.

[2] Similar principle for higher order derivatives

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So far $\|(\nabla^{\perp}_{\partial_s})^m \vec{\kappa}\|_{L^2} \leq c(m, \delta)$, $t \in [\delta, T_{\max})$, $m \in \mathbb{N}$.
First issue: get rid of the projection: use that

$$\partial_s \vec{\kappa} = \nabla^{\perp}_{\partial_s} \vec{\kappa} - |\vec{\kappa}|^2 \partial_s f \,.$$

and similarly $\partial_s^m \vec{\kappa} = (\nabla^{\perp}_{\partial_s})^m \vec{\kappa} + \text{l.o.t.}$

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unif. bdd

Second issue: get rid of ∂_s Follows from

$$\partial_t |\partial_x f| = -\langle \vec{\kappa}, \partial_t f \rangle |\partial_x f|,$$

and the regularity of the initial datum[2].

[2]Similar principle for higher order derivatives

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For $\lambda = 0$:

$$\frac{d}{dt}\mathcal{L}[f] = \int_{\mathbb{S}^1} \langle \vec{\kappa}, \partial_f t \rangle \leq C.$$

$$\mathcal{L}(\gamma(t)) = \mathcal{L}(\gamma(0)) + \int_0^t \mathcal{L}(\gamma(s))\, ds$$

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$W^{m,2} \hookrightarrow C^{m-1,\alpha}$, $\alpha \in (0,1/2)$

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Together with control of $\|f\|_\infty$ by

$$f(t,x) = f(0,x) + \int_0^t \partial_t f(s,x)\, ds,$$

we get uniform bounds in $W^{2,m} \subset C^{m-1,\alpha}$.

Subconvergence

Lemma

For $\lambda > 0$, as $t_i \to \infty$ there exist $p_i \in \mathbb{R}^n$ such the curves $f(\cdot, t_{i_k}) - p_{i_k}$ converge, when reparametrized by constant speed, to a critical point of $\mathcal{E}_\lambda$.

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Let $(f(\cdot, t_i))_i$ be parametrised by constant speed on $[0, 1]$. Since $\lambda > 0$, the length of $(f(\cdot, t_i))_i$ are uniformly bounded by $L < \infty$. Let $(p_i)_i \in \mathbb{R}^n$ s.t.

$$\hat{f}(\cdot, t_i) = f(\cdot, t_i) - p_i$$

go through the point $0 \in \mathbb{R}^n$.

By construction $(\hat{f}(t_i))_i$ are uniformly bounded in C^m and stay in a compact subset of $\mathbb{R}^n$. Hence there exists a subsequence $(t_{i_j})_j$ and $\hat{f}$ smooth such that $\hat{f}(t_{i_j}) \to \hat{f}$.

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$$u(t) = \int_I |\partial_t f|^2 ds = -\frac{d}{dt}\mathcal{E}_\lambda(f),$$

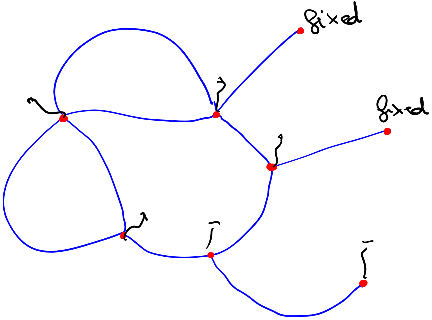
we have that $u \in L^1(0,\infty)$. Moreover, $|u'(t)|$ is uniformly bounded. It follows that

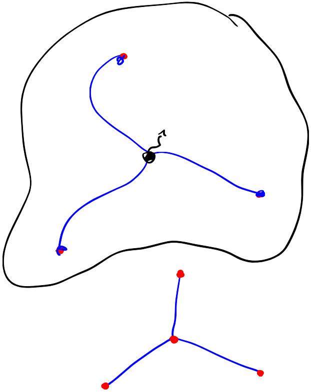
$$\lim_{t\to\infty} u(t) = 0.$$

Thus $\hat{f}$ is a critical point for $\mathcal{E}_\lambda$.

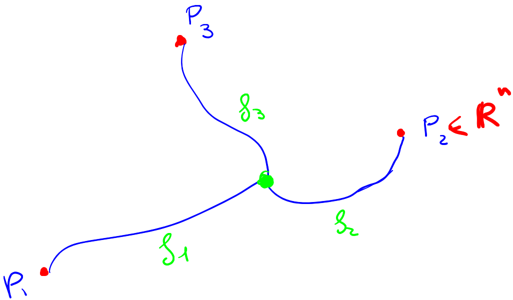
Networks

Goal: Understand the elastic flow of networks.





The easiest situation: star-shaped network



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Consider three smooth regular curves $f_i : I \to \mathbb{R}^n$, $I = [0,1]$, $i = 1,2,3$, such that

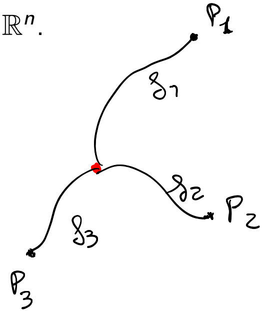
1. The end-points are fixed:

$$f_1(1) = P_1,\ f_2(1) = P_2,\ f_3(1) = P_3,$$

   with given distinct points P_i, $i = 1,2,3$, in $\mathbb{R}^n$.

2. The curves start at the same point

$$f_1(0) = f_2(0) = f_3(0)\,.$$



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For $\lambda > 0$, the elastic energy of the network is

$$\mathcal{E}_\lambda(\Gamma) = \sum_{i=1}^{3} \mathcal{E}_\lambda(f_i)\,.$$

$\frac{1}{2}\int |\kappa(f_i)|^2 + \lambda \mathcal{L}(f_i)$

First idea

How to model the flow of a network?

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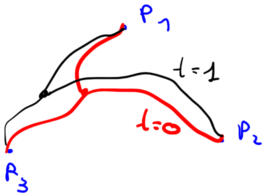
Let $f_{0,i}$ be smooth enough, regular such that $\{f_{0,1}, f_{0,2}, f_{0,3}\}$ is a star-shaped network, and consider

$$\begin{cases} \partial_t f_i = -(\nabla^{\perp}_{\partial_s})^2 \vec{\kappa}_i - \frac{1}{2}|\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda \vec{\kappa}_i, & \text{on } (0,T)\times[0,1], \\ f_i(t,0) = f_j(t,0) & \text{on } (0,T),\, i,j = 1,2,3, \\ f_i(t,1) = P_i & \text{on } (0,T),\, i = 1,2,3. \end{cases}$$

and $f(0,x) = f_0(x)$.



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Some problems

1. Number of boundary conditions

First idea

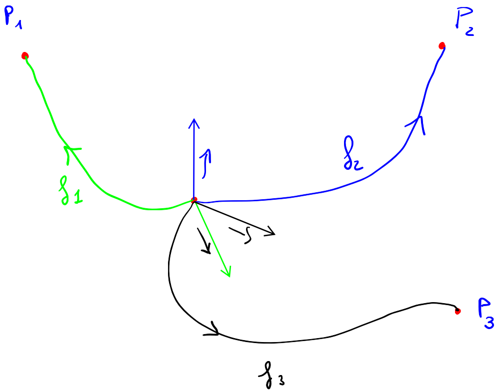
How to model the flow of a network?
Let $f_{0,i}$ be smooth enough, regular such that $\{f_{0,1}, f_{0,2}, f_{0,3}\}$ is a star-shaped network, and consider

$$\begin{cases} \partial_t f_i = -(\nabla^{\perp}_{\partial_s})^2 \vec{\kappa}_i - \frac{1}{2}|\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda \vec{\kappa}_i, & \text{on } (0,T) \times [0,1], \\ f_i(t,0) = f_j(t,0) & \text{on } (0,T), i,j = 1,2,3, \\ f_i(t,1) = P_i & \text{on } (0,T), i = 1,2,3. \end{cases}$$

and $f(0,x) = f_0(x)$.
Some problems

1. Number of boundary conditions
2. Does the solution (if it exists) describe a network? That is, what about the Topology?

Situation



$\gamma_i : [0,1] \longrightarrow \mathbb{R}^n$

$\partial_t \gamma_i = \underbrace{-\nabla_{L^2} \mathcal{E}(\gamma)}_{\text{normal}} + \varphi_i \partial_s \gamma_i$

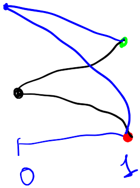
Boundary conditions: a toy computation

Consider

$$E(u,v) = \frac{1}{2}\int_0^1 (u_{xx})^2 + (v_{xx})^2 dx$$

defined for $u, v \in W^{2,2}(0,1)$ such that

$$u(0) = v(0) \text{ and } u(1) = 0, v(1) = 1.$$



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First variation:

$$\frac{d}{dt}E(u+t\varphi, v+t\psi)\Big|_{t=0} = \frac{1}{2}\frac{d}{dt}\int_0^1 (u_{xx}+t\varphi_{xx})^2 + (v_{xx}+t\psi_{xx})^2 dx\Big|_{t=0}$$

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$$\begin{aligned}\frac{d}{dt}E(u+t\varphi, v+t\psi)\Big|_{t=0} &= \int_0^1 u_{xx}\varphi_{xx} + v_{xx}\psi_{xx} dx \\ &= \underbrace{u_{xx}\varphi_x|_0^1} - \underbrace{u_{xxx}\varphi|_0^1} + v_{xx}\psi_x|_0^1 - v_{xxx}\psi|_0^1 \\ &\quad + \int_0^1 u_{xxxx}\varphi + v_{xxxx}\psi dx\end{aligned}$$

Boundary conditions: a toy computation...

For $u, v \in W^{2,2}(0,1)$ with $u(0) = v(0)$, $u(1) = 0, v(1) = 1$,

$$E(u,v) = \frac{1}{2}\int_0^1 (u_{xx})^2 + (v_{xx})^2 dx$$

$(u + t\varphi)(1) = 0$

$\Rightarrow \varphi(1) = 0$

$\Rightarrow \psi(1) = 0$

First variation:

$$\frac{d}{dt}E(u+t\varphi, v+t\psi)\Big|_{t=0} = \int_0^1 u_{xx}\varphi_{xx} + v_{xx}\psi_{xx} dx$$

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for (φ, ψ) admissible variation.

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for (φ, ψ) admissible variation. That is,

$$\varphi(1) = 0 = \psi(1) \text{ und } \varphi(0) = \psi(0).$$

Natural boundary conditions:

$$u_{xx}(0) = u_{xx}(1) = 0 = v_{xx}(0) = v_{xx}(1) \text{ und } u_{xxx}(0) = v_{xxx}(0).$$

Statement of the problem

Find curves $f_i : [0,T) \times [0,1] \to \mathbb{R}^n$ solution to

$$\partial_t f_i = -\nabla_s^2 \vec{\kappa}_i - \frac{1}{2}|\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda_i \vec{\kappa} + \varphi_i \partial_s f_i \text{ on } (0,T) \times I,$$

for $i = 1,2,3$ with boundary conditions

$$\begin{cases} f_i(t,1) = P_i, & \text{for } t \in (0,T), i = 1,2,3, \\ \vec{\kappa}_i(t,1) = 0 = \vec{\kappa}_i(t,0) & \text{for } t \in (0,T), \ i = 1,2,3, \\ f_1(t,0) = f_2(t,0) = f_3(t,0) & \text{for } t \in (0,T), \\ \text{and } \sum_{i=1}^3 (\nabla_s \vec{\kappa}_i(t,0) - \lambda_i \partial_s f_i(t,0)) = 0 & \text{for } t \in (0,T), \end{cases}$$

and initial value

$$f_1(t=0,\cdot) = f_{i,0} \text{ for } i = 1,2,3, \text{ in } [0,1].$$

Better statement of the problem

Find curves $f_i : [0, T) \times [0, 1] \to \mathbb{R}^n$ and tangential components $\varphi_i : [0, T) \times [0, 1] \to \mathbb{R}$ solution to

$$\partial_t f_i = -\nabla_s^2 \vec{\kappa}_i - \frac{1}{2}|\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda_i \vec{\kappa}_i + \varphi_i \partial_s f_i \text{ on } (0, T) \times I,$$

for $i = 1, 2, 3$ with boundary condition as before and initial value as before.

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Joint work with C.-C. Lin and P. Pozzi.
Also: H. Garcke, J. Mendez and A. Pluda.

Short time existence

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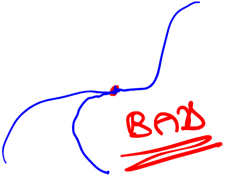
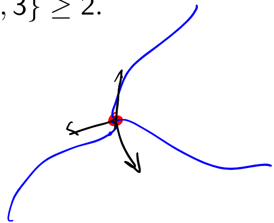
(x=0)

$$\mathrm{Span}\{\partial_s f_{0,i} : i = 1, 2, 3\} \geq 2.$$





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4. With Banach Fixed Point Theorem: a solution of the quasilinear problem exists in a short interval of time.

Long time existence

Idea: Show that all the norms of the solution remain bounded and then extend the solution. Non-collinearity condition needed at $T_{\max}$.