On the elastic flow

curves, networks and Willmore flow of tori of revolution

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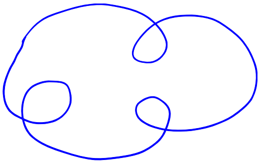
Table of contents

Problem

Consider a springy wire which ends are joint smoothly. What happens when we release it?



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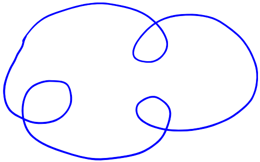

The wire tries to reduce ist bending (or elastic) energy. This we want to model.

Curvature of a curve

For $f : \mathbb{S}^1 \to \mathbb{R}^n$ regular smooth curve, the length is

$$L(f) = \int_{\mathbb{S}^1} |\partial_x f(x)|\, dx = \int_{\mathbb{S}^1} ds\,,$$

the tangential vector is given by

$$\vec{T}(x) = \frac{1}{|\partial_x f|}\partial_x f = \partial_s f\,, \qquad \frac{1}{|\partial_x f|}\partial_x = \partial_s$$

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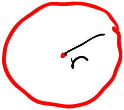
$$\vec{T}(x) = \frac{1}{|\partial_x f|} \partial_x f = \partial_s f \ ,$$

and the curvature is given by

$$\vec{\kappa}(x) = \partial_s^2 f(x) = \frac{1}{|\partial_x f|} \partial_x \vec{T}(x) \,.$$

Example



For

$$f(x) = \begin{pmatrix} r\cos(x) \\ r\sin(x) \end{pmatrix}$$

then

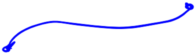
$$\partial_x f(x) = \begin{pmatrix} -r\sin(x) \\ r\cos(x) \end{pmatrix}, \quad |\partial_x f(x)| = r \text{ and } \partial_s f(x) = \begin{pmatrix} -\sin(x) \\ \cos(x) \end{pmatrix},$$

and

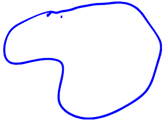
$$\vec{\kappa}(x) = \frac{1}{r}\begin{pmatrix} -\cos(x) \\ -\sin(x) \end{pmatrix}.$$

The elastic energy

$\gamma : [0,1] \longrightarrow \mathbb{R}^n$



In the Bernoulli model of elasticity, the elastic energy of an elastic rod, described by a curve $f : \mathbb{S}^1 \to \mathbb{R}^n$, is given by



$$\int_{\mathbb{S}^1} |\vec{\kappa}|^2 \, ds$$

with s the arc-length parameter ($ds = |\partial_x f| dx$), and $\vec{\kappa}$ the curvature.

The elastic energy on a Riemannian manifold

Let (M^n, g) be a Riem. manifold. Then for a closed regular curve $f : \mathbb{S}^1 \to M$ we define its elastic energy as

$$\mathcal{E}(f) = \frac{1}{2} \int_{\mathbb{S}^1} |\vec{\kappa}|_g^2 \, \mathrm{d}s \,,$$

with $\mathrm{d}s = |\partial_x f|_g \, \mathrm{d}x$ and the $\vec{\kappa}$ the geodesic curvature, $\vec{\kappa} = \nabla_{\partial_s f} \partial_s f$.

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In this talk $M = \mathbb{R}^n$.

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In this talk $M = \mathbb{R}^n$.
Interesting: the case $M = \mathbb{H}^2$. The elastic energy of curves in $\mathbb{H}^2$ is related to Willmore surfaces of revolution. See third talk.

What about minimizers?

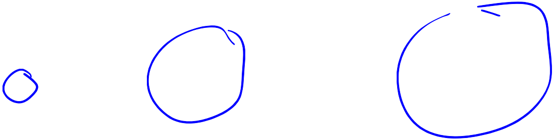
What about minimizers?

This problem is in general not well posed. Take $M = \mathbb{R}^n$ and f_n a regular parametrization of the circle of radius n. Then

$$|\vec{\kappa}| = \frac{1}{n} \text{ and } \mathcal{E}(f_n) = \frac{1}{n^2} 2\pi n \to 0\,,$$

$\mathcal{E}(8) = \int |\kappa|^2 ds$

and every regular closed curve f satisfies $\mathcal{E}(f) > 0$.



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Solution Fix or penalize the length. Actually due to rescaling properties these two minimization problems are (sort of) 'equivalent'.

$\mathcal{E}_\lambda(\gamma) = \mathcal{E}(\gamma) + \lambda \mathcal{L}(\gamma)$

Scaling behavior of the energy

For $\lambda > 0$ consider

$$\mathcal{E}_\lambda(f) = \frac{1}{2}\int_{\mathbb{S}^1} |\vec{\kappa}|^2 \, \mathrm{d}s + \lambda \int_{\mathbb{S}^1} 1 \, \mathrm{d}s = \mathcal{E}(f) + \lambda L(f)\,.$$

Scaling behavior of the energy

For $\lambda > 0$ consider

$$\mathcal{E}_\lambda(f) = \frac{1}{2}\int_{\mathbb{S}^1} \underbrace{|\vec{\kappa}|^2}_{\frac{1}{R^2}} \underbrace{\mathrm{d}s}_{R} + \lambda \int_{\mathbb{S}^1} 1\,\mathrm{d}s = \mathcal{E}(f) + \lambda L(f)\,.$$

For $R > 0$

$$L(Rf) = RL(f) \text{ and } \mathcal{E}(Rf) = \frac{1}{R}\mathcal{E}(f)\,,$$

$\tilde{g} = R g$ $|\tilde{\kappa}| = \frac{1}{R}|\kappa|$

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$$\mathcal{E}_\lambda(\lambda^{\frac{-1}{2}} f) = \lambda^{\frac{1}{2}}\mathcal{E}(f) + \lambda\lambda^{\frac{-1}{2}} L(f) = \lambda^{\frac{1}{2}}\mathcal{E}_1(f)\,.$$

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$\lambda = 1$

Corollary If f minimize $\mathcal{E}_1$ then $\lambda^{\frac{-1}{2}} f$ minimizes $\mathcal{E}_\lambda$.
I.e. w.l.o.g. $\lambda = 1$.

Equivalence of the two problems

$\mathcal{E}(\gamma) + L(\gamma)$
$\|$

Lemma If f minimizes $\mathcal{E}_1$, then for every $L > 0$ there exists an $R > 0$ s.t. Rf minimizes $\mathcal{E}$ over curves with fixed length L and vice versa.

Equivalence of the two problems

$\mathcal{E}_1 = \mathcal{E} + \lambda L$

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$$\underbrace{\mathcal{E}_1(f)} \overset{\text{Def.}}{=} \mathcal{E}(f)+\cancel{L(f)} \underset{\uparrow}{\leq} \mathcal{E}(\rho\tilde{f})+\cancel{L(\rho\tilde{f})} = \underbrace{\mathcal{E}_1(\rho\tilde{f})}$$

f min $\mathcal{E}_1$

$\mathcal{E}_1(\rho f) \geq \mathcal{E}_1(f)$

and hence $\underbrace{\mathcal{E}(f) \leq \mathcal{E}(\rho\tilde{f})}$

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$$\mathcal{E}_1(f) = \mathcal{E}(f) + L(f) \leq \mathcal{E}(\rho\tilde{f}) + L(\rho\tilde{f}) = \mathcal{E}_1(\rho\tilde{f})$$

and hence $\mathcal{E}(f) \leq \mathcal{E}(\rho\tilde{f})$ so that

$$\mathcal{E}(\rho^{-1}f) = \rho\mathcal{E}(f) \leq \rho\mathcal{E}(\rho\tilde{f}) = \mathcal{E}(\tilde{f}).$$

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'$\Leftarrow$' Exercise.

Question

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Hence

$$2\pi \leq \mathcal{E}(\gamma) \,,$$

i.e. 2π is a lower bound and it is attained by a circle of radius one.

Critical points

$\vec{k} = \partial_s 8$

Critical points of

$$\mathcal{E}(f) = \frac{1}{2}\int_{\mathbb{S}^1} |\vec{\kappa}|^2 \, \mathrm{d}s \,,$$

are called (free) elastica and satisfy the equation

$= 0$

$$\nabla_{L^2}\mathcal{E}(f) = (\nabla^{\perp}_{\partial_s})^2 \vec{\kappa} + \frac{1}{2}|\vec{\kappa}|^2\vec{\kappa} \; \cancel{(+ S_0 \vec{\kappa})} = 0 \,,$$

where $\nabla^{\perp}_{\partial_s}\vec{\kappa} = \partial_s\vec{\kappa} - \langle \partial_s\vec{\kappa}, \partial_s f\rangle \partial_s f$: a geometric equation.

$\partial_s \vec{k}$

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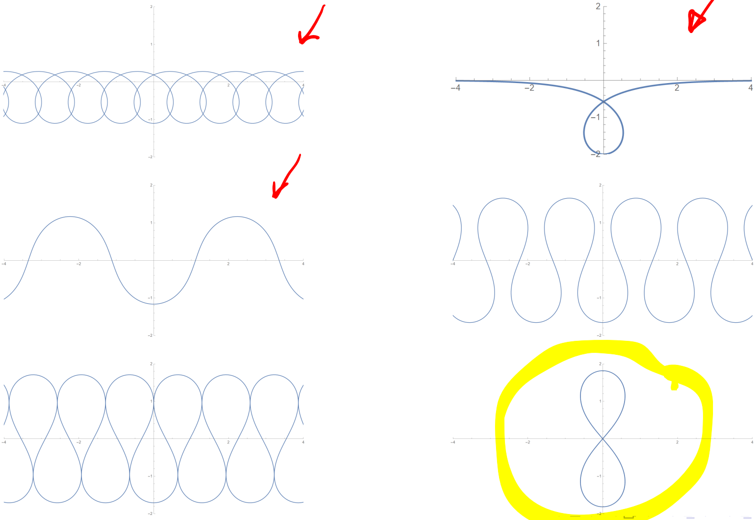
If we penalize the length (i.e. consider $\mathcal{E}(f)+\lambda\int_{\mathbb{S}^1} 1\,\mathrm{d}s$) or prescribe a fix length, then

$$\nabla_{L^2}\mathcal{E}(f) = (\nabla^{\perp}_{\partial_s})^2\vec{\kappa} + \frac{1}{2}|\vec{\kappa}|^2\vec{\kappa}\ (+S_0\vec{\kappa}) - \lambda\vec{\kappa} = 0\,.$$

Examples of elastica: Langer-Singer

Thanks to Fabian!

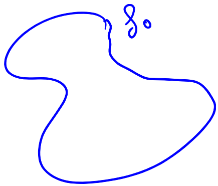


Gradient Flow

Another approach to find critical points: L^2-gradient flow.

$$\begin{cases} \partial_t f = -\nabla_{L^2}\mathcal{E}_\lambda(f), & \text{in } \mathbb{S}^1 \times (0,T), \\ f(x,0) = f_0(x) & \text{for } x \in \mathbb{S}^1, \end{cases}$$

with $f_0 : \mathbb{S}^1 \to M$ given.



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$\lambda \geq 0$

with $f_0 : \mathbb{S}^1 \to M$ given.

Notice that if $t \to f(t)$ is solution, then

$$\frac{d}{dt}\mathcal{E}_\lambda(f(t)) = \mathcal{E}'_\lambda(f(t))\partial_t f = \langle \nabla_{L^2}\mathcal{E}_\lambda(f(t)), \partial_t f\rangle_{L^2}$$

$$= -\|\nabla_{L^2}\mathcal{E}_\lambda\|_{L^2} \leq 0$$

$$\mathcal{E}_\lambda(\gamma) = \frac{1}{2}\int |\kappa|^2 ds + \lambda \int ds$$

$$\frac{1}{2}\|\kappa\|_{L^2}^2 \leq \mathcal{E}_\lambda(\gamma(t)) \leq \mathcal{E}_\lambda(\gamma_0)$$

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with $f_0 : \mathbb{S}^1 \to M$ given.

Notice that

$$\frac{d}{dt}\mathcal{E}_\lambda(f(t)) = \langle \nabla \mathcal{E}_\lambda(f(t)), \partial_t f \rangle_{L^2} = -\|\nabla \mathcal{E}_\lambda(f(t))\|^2 \leq 0.$$

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Today: classical arguments to treat this 4^{th}-order equation. Dziuk-Kuwert-Schätzle, Coworkers: C.-C. Lin, M. Müller, P. Pozzi, F. Rupp, A. Spener

Problem

For $f_0 : \mathbb{S}^1 \to \mathbb{R}^n$ a smooth, regular and closed curve and $\lambda \geq 0$ study the existence of a smooth solution $f : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^n$ of

$$\begin{cases} \partial_t f = -(\nabla^{\perp}_{\partial_s})^2 \vec{\kappa} - \frac{1}{2}|\vec{\kappa}|^2 \vec{\kappa} + \lambda \vec{\kappa}, & \text{in } \mathbb{S}^1 \times (0, T), \\ f(x, 0) = f_0(x) & \text{for } x \in \mathbb{S}^1 . \end{cases}$$

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Questions

1. Short time Existence.

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2. Long time Existence.

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3. Asymptotic behavior: Løjasiewicz-Simon inequality: see also the talk of Kohout and Rupp.

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Next talks: also networks and Willmore flow of tori.

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$\vec{k} = \partial_s^2 f$

Features

1. Fourth order equation.

Problem

For $f_0 : \mathbb{S}^1 \to \mathbb{R}^n$ a smooth, regular and closed curve and $\lambda \geq 0$ study the existence of a smooth solution $f : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^n$ of

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Features

1. Fourth order equation.
2. Parabolicity is a problem: $\nabla^\perp_{\partial_s}$ has a projection.

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2. Parabolicity is a problem: $\nabla_{\partial_s}^{\perp}$ has a projection.
3. We expect long-time existence: different from the curve shortening flow $\partial_t f = \vec{\kappa}$.

Problem

For $f_0 : \mathbb{S}^1 \to \mathbb{R}^n$ a smooth, regular and closed curve and $\lambda \geq 0$ study the existence of a smooth solution $f : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^n$ of

$$\begin{cases} \partial_t f = -(\nabla_{\partial_s}^{\perp})^2 \vec{\kappa} - \frac{1}{2}|\vec{\kappa}|^2 \vec{\kappa} + \lambda \vec{\kappa}, & \text{in } \mathbb{S}^1 \times (0, T), \\ f(x,0) = f_0(x) & \text{for } x \in \mathbb{S}^1 . \end{cases}$$

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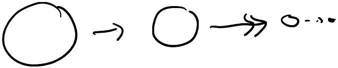
1. Fourth order equation.
2. Parabolicity is a problem: $\nabla_{\partial_s}^{\perp}$ has a projection.
3. We expect long-time existence: different from the curve shortening flow $\partial_t f = \vec{\kappa}$. Ansatz: circular solution, then $\partial_t f = -\frac{1}{r^2} f$.



Ansatz

Look for a family of circles as a solution, i.e.

$$f(t,x) = r(t)\begin{pmatrix}\cos x\\ \sin x\end{pmatrix}.$$

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$|\kappa|^2\kappa$

Flow equation becomes

$$r'(t)\begin{pmatrix}\cos x\\ \sin x\end{pmatrix} = 0 - \frac{1}{2}\frac{1}{(r(t))^3}\begin{pmatrix}-\cos x\\ -\sin x\end{pmatrix} + \overbrace{\frac{\lambda}{r(t)}\begin{pmatrix}-\cos x\\ -\sin x\end{pmatrix}}^{\lambda\kappa}$$

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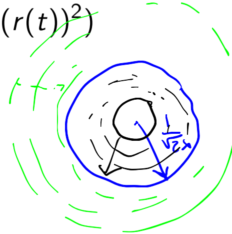

if $\lambda = 0$: then $r(t) \to \infty$ for $t \to \infty$;
if $\lambda > 0$: then $r(t) \to \frac{1}{\sqrt{2\lambda}}$ for $t \to \infty$;

Short time result

Let f_0 be smooth enough, regular and closed. Then $\exists T > 0$ and a smooth solution $f : (0, T) \times \mathbb{S}^1 \to \mathbb{R}^n$ continuous up to $t = 0$ to

$$\partial_t f = -(\nabla^{\perp}_{\partial_s})^2 \vec{\kappa} - \frac{1}{2}|\vec{\kappa}|^2 \vec{\kappa} + \lambda \vec{\kappa},$$

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$$\begin{aligned}\partial_t f = & -\frac{\partial_x^4 f}{|\partial_x f|^4} + 6\langle \partial_x^2 f, \partial_x f\rangle \frac{\partial_x^3 f}{|\partial_x f|^6} + \frac{\partial_x^2 f}{|\partial_x f|^2}\Big(\frac{5}{2}\frac{|\partial_x^2 f|^2}{|\partial_x f|^4} + \dots\Big) \\ & - \frac{\partial_x f}{|\partial_x f|}\Big[-\langle \frac{\partial_x^4 f}{|\partial_x f|^5}, \partial_x f\rangle + 10 \frac{\langle \partial_x^2 f, \partial_x f\rangle}{|\partial_x f|^7}\langle \partial_x^3 f, \partial_x f\rangle + \dots \Big].\end{aligned}$$

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1. Add a tangential component to make it parabolic.
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Method

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2. Linearize the problem: show that the linear problem is well posed.
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4. Adjust the solution via a diffeomorphism to get rid of the tangential component you added.

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The solution exists for all times, that is $T_{max} = \infty$.

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$T_{max} < \infty$

Idea of the proof Show that in finite time any norm of the solution is bounded. Hence it can always be extended.

Main estimate $\|\partial_s^m \vec{\kappa}\|_{L^\infty} \leq c(m,\delta)$, $t \in [\delta, T_{\max})$, $m \in \mathbb{N}$.

$T_{max} < \infty$

$L(\gamma) < \infty$

$\frac{d}{dt} \mathcal{L}(\gamma) = \nabla^s k$

$= \int \langle k, \partial_t^m \gamma \rangle ds \leq C$

$\mathcal{E}(\gamma) = \frac{1}{2} \|\vec{k}\|_{L^2}^2 \leq C.$

$\lambda > 0$: $\underbrace{\mathcal{E}_\lambda(\gamma(t))}_{= \mathcal{E}(\gamma) + \lambda \mathcal{L}(\gamma)} \leq \mathcal{E}_\lambda(\gamma_0) < \infty$

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$\geq \lambda \mathcal{L}(\gamma)$

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Then, since the initial datum is regular in finite time the parametrization can be controlled. One derives

$$\|\partial_x^m \vec{\kappa}\|_{L^\infty} \leq c(m,\delta), \; t \in [\delta, T_{\max}) \, .$$

$$\partial_t \underbrace{|\partial_x \gamma|} = \ldots \nabla_s^m k$$

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$$\begin{aligned} \partial_s \vec{\kappa} &= \nabla^{\perp}_{\partial_s}\vec{\kappa} + \langle \partial_s \vec{\kappa}, \partial_s f\rangle_g \partial_s f \quad (1) \qquad (2)\\ &= \nabla^{\perp}_{\partial_s}\vec{\kappa} + \partial_s\langle \vec{\kappa}, \partial_s f\rangle_g \partial_s f - \langle \vec{\kappa}, \partial_s\partial_s f\rangle_g \partial_s f \\ &= \nabla^{\perp}_{\partial_s}\vec{\kappa} - |\vec{\kappa}|^2 \partial_s f\,. \end{aligned}$$

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$$\begin{aligned}
\partial_s \vec{\kappa} &= \nabla^{\perp}_{\partial_s} \vec{\kappa} + \langle \partial_s \vec{\kappa}, \partial_s f\rangle_g \partial_s f \\
&= \nabla^{\perp}_{\partial_s} \vec{\kappa} + \partial_s \langle \vec{\kappa}, \partial_s f\rangle_g \partial_s f - \langle \vec{\kappa}, \partial_s \partial_s f\rangle_g \partial_s f \\
&= \nabla^{\perp}_{\partial_s} \vec{\kappa} - |\vec{\kappa}|^2 \partial_s f\,.
\end{aligned}$$

Similarly for higher m one has $\nabla^m_{\partial_s} \vec{\kappa} = (\nabla^{\perp}_{\partial_s})^m \vec{\kappa} + \text{l.o.t.}$