UNIVERSITY OF COPENHAGEN

CARLSBERGFONDET

European Research Council

Workshop on real applied algebra

Book of abstracts

March 23–25, 2026

Monday 23rd of March 2026

- 9.00 - 9.50 (Invited talk): **Irem Portakal** - MPI for Mathematics in the Sciences (Leipzig).

  The Nash equilibrium scheme

  Abstract: In this talk, we explore the geometry underlying totally mixed Nash equilibria for n-player games. We model these equilibria using vector bundles and introduce the Nash equilibrium scheme, which parametrizes all totally mixed equilibria of a given game. Depending on the format of the game, we define two key geometric objects: the Nash discriminant and the Nash resultant varieties. These capture games whose Nash equilibrium schemes exhibit unexpected behavior - such as being nonreduced or having positive-dimensional components. No prior background in game theory will be assumed, and we will illustrate the main ideas with concrete examples. This is joint work with Hirotachi Abo and Luca Sodomaco.

- 9.50 - 10.15 (Contributed talk): **Roser Homs** - Universitat Politècnica de Catalunya (Barcelona).

  Maximum likelihood thresholds for colored Gaussian graphical models

  Abstract: Colored Gaussian graphical models are statistical models arising from undirected graphs with a coloring in its vertices and edges. We study maximum likelihood thresholds for these models: the minimum number of observations that ensure existence (with either probability one or strictly positive probability) of the maximum likelihood estimator. We extend results for ML thresholds to the colored setting, implement algorithms that exploit the underlying algebraic geometry and compute the thresholds for certain families of graphs. Based on joint work with Olga Kuznetsova, Bernadette J. Stolz, Aida Maraj, Danai Deligeorgaky, Joe Johnson, and Bryson Kagy.

- 10.15 - 10.40 (Contributed talk): **Niharika Chakrabarty Paul** - MPI for Mathematics in the Sciences (Leipzig).

  Torus actions on matrix Schubert and Kazhdan-Lusztig varieties and their links to statistical models

  Abstract: We investigate the toric geometry of two families of generalised determinantal varieties arising from permutations: Matrix Schubert varieties ($\overline{X_w}$) and Kazhdan-Lusztig varieties ($\mathcal{N}_{v,w}$). Matrix Schubert varieties can be written as $\overline{X_w} = Y_w \times \mathbb{C}^d$, where d is maximal. We are especially interested in the structure and complexity of these varieties Y_w and $\mathcal{N}_{v,w}$ under the so-called usual torus actions. Finally, we consider the links between these determinantal varieties and two classes of statistical models; namely conditional independence and quasi-independence models. This is joint work with Elke Neuhaus and Irem Portakal.

- 11.00 - 11.50 (Invited talk): **Jie Wang** - AMSS-CAS (Beijing).

  Scalable ground-state certification of quantum spin systems via structured noncommutative polynomial optimization

  Abstract: A fundamental challenge in quantum physics is determining the ground-state properties of many-body systems. For quantum ½-spin systems, this problem can be formulated as a noncommutative polynomial optimization problem and addressed using a hierarchy of semidefinite programming relaxations. However, this approach typically suffers from severe scalability issues, limiting its applicability to small-scale systems. In this talk, we demonstrate that leveraging the inherent structures of the system can significantly mitigate these scalability challenges.

- 13.20 - 14.10 (Invited talk): **Matías Bender** - INRIA Saclay-CMAP (Paris-Saclay).

  Certificates of nonnegativity of polynomials

  Abstract: We study the problem of certifying the nonnegativity of univariate and multivariate polynomials with rational coefficients. We present new certificates based on representing the input polynomial as a sum of squares modulo specially constructed ideals, which guarantees the desired nonnegativity property. In addition, we introduce new algorithms to compute such certificates. These algorithms have lower complexity than previously known methods and are applicable to any polynomial, without assumptions on radicality or the finiteness of critical points. They can also exploit the sparsity structure of the input polynomial. In the univariate case, we uncover a connection between sums of squares and Karlin's theory of T-systems. This talk is based on joint work with Philipp di Dio, Khazhgali Kozhasov, Elias Tsigaridas, and Chaoping Zhu.

- 14.10 - 14.35 (Contributed talk): **Alexander Zenkovich** - École Normale Supérieure (Paris).

  A new spherical criterion for unconstrained polynomial optimization

  Abstract: We introduce a criterion of nonnegativity for polynomials with real coefficients, that makes no assumptions on the input. The vast majority of current approaches, to identify or certify, the nonnegativity of a polynomial $f \in \mathbb{R}[\mathbf{X}]$ has to assume that the infimum is attained and/or the gradient ideal, that is the system of the derivatives of f, is zero-dimensional. We overcome both obstacles by, first homogenizing the polynomial and then, perform a *non-homogeneous* pertubation of the *homogeneous* gradient ideal; the points of the latter correspond to the critical points of the (homogeneous) polynomial on a unit sphere. We emphasize the simplicity of the perturbation that does not alter, neither the degree, nor the bitsize (up to logarithmic factors) of the input. We use the criterion to introduce a certificate of nonnegativity, which is both perfectly complete (it certifies every nonnegative polynomial) and perfectly sound (it correctly rejects any polynomial that is not nonnegative). In addition, we apply the criterion to the unconstrained polynomial optimization problem, where we present an SOS hierarchy that converges to the optimum value.

- 15.10 - 15.35 (Contributed talk): **Jaewoo Jung** - Daegu Gyeongbuk Institute of Science and Technology (Daegu).

  On quadratic persistence and Pythagoras numbers of totally real projective varieties

  Abstract: The Pythagoras number of a real projective variety measures the minimal number of squares required to express sums of squares of linear forms in its coordinate ring. Building on the study of bounds for this invariant in terms of algebraic invariants, Blekherman, Sinn, Smith, and Velasco classified totally real projective varieties attaining the minimal and next-to-minimal Pythagoras numbers, under the additional assumption that the variety is arithmetically Cohen–Macaulay (aCM). In this work, we extend their classification to the non–aCM setting by replacing the aCM hypothesis with a degree condition, thereby obtaining a complete characterization of totally real varieties with next-to-minimal Pythagoras numbers whenever the degree condition is satisfied. In addition, we examine two families of curves not covered by our main theorem—curves of maximal regularity and linearly normal smooth curves of genus three—and determine their Pythagoras numbers.

- 15.35 - 16.00 (Contributed talk): **Nico Lorenz** - Ruhr-Universität Bochum (Bochum).

  Pythagoras number and supreme torsion forms

  Abstract: Let F be a formally real field. Associated object like its orderings have a strong influence on the theory of quadratic forms over F. They determine e.g. the prime ideals and torsion elements in the Witt ring $W(F)$, the ring classifying quadratic forms over F. Further the *Pythagoras number* of F, i.e. the least integer n such that every sum of squares in F is already a sum of n squares in F, and the size of the *square class group* F^*/F^{*2} interact in interesting ways with each other and with orderings. In this talk we survey classical results about these arithmetical invariants and use the concept of *supreme torsion forms* to construct examples of real fields with prescribed Pythagoras number in which there are in a certain sense as few squares as possible.

- 16.00 - 16.30 (Contributed talk): **Tomasz Kowalczyk** - Jagiellonian University (Krakow).

  Sums of squares of regular functions on real rational surfaces

  Abstract: We study the Pythagoras number (i.e. the smallest positive integer g, such that any sum of squares is a sum of at most g squares) of the ring of regular functions on curves and cylinders (products of a curve with a line). We then apply these results to the case of (uniformly) rational surfaces. In particular, we show the following. Let X be a nonsingular rational surface over the reals and f be a polynomial function which is a sum of squares in the field of rational functions on X. Then there exists a strictly positive polynomial function h on X such that hf is a sum of at most 12 squares of polynomial functions.

Tuesday 24th of March 2026

- 9.00 - 9.50 (Invited talk): **Khazhgali Kozhasov** - Université Côte d'Azur (Nice).

  Stubborn polynomials

  Abstract: The relationship between nonnegative polynomials and sums of squares is a classical topic in real algebraic geometry going back to Hilbert. By Artin's solution (1927) of Hilbert's 17th problem, for every nonnegative polynomial P there is a real polynomial Q such that PQ^2 is a sum of squares of real polynomials. Later many authors considered different variations of this problem. For example, Scheiderer (2012) showed that if both P and Q are positive definite (that is, have no real projective zeros), then PQ^k is a sum of squares for all sufficiently large k. In particular, this holds for P^k, whenever P is positive definite. However, if P has real zeros, it might fail to have this property. In fact, it could be that P^k is NOT a sum of squares for all odd powers k. We call such nonnegative polynomials stubborn. Many classical examples of nonnegative polynomials are now known to be stubborn, among them are Motzkin and Robinson polynomials. I will give an overview of the state-of-the-art results on stubborn polynomials obtained in joint works with Blekherman and Reznick and with Baldi, Blekherman, Plaumann, Reznick and Sinn.

- 9.50 - 10.15 (Contributed talk): **Francesco Maria Mascarin** - MPI for Mathematics in the Sciences (Leipzig).

  Lissajous varieties

  Abstract: We introduce Lissajous varieties as affine algebraic varieties parametrized by sine and cosine functions that generalize algebraic Lissajous curves in the plane to higher dimension. We show that the degree of these varieties equals the volume of a corresponding polytope, up to a combinatorial factor, and that their defining equations arise from rank constraints on an associated polynomial matrix. We discuss their applications in dynamical systems, in particular the Kuramoto model and then explore their connections with convex optimization and Lissajous discriminants. Finally, in the context of semidefinite programming, we highlight the crucial role these varieties play in capturing the algebraic boundary of the elliptope for certain graph classes. This is based on joint works with Monique Laurent and Simon Telen.

- 10.15 - 10.40 (Contributed talk): **Oskar Henriksson** - MPI of of Molecular Cell Biology and Genetics (Dresden).

  Computing discriminant complements using pseudo-witness sets

  Abstract: Many problems in applied algebraic geometry boil down to understanding what possible geometries the real solution set of a parametric polynomial system can take as the parameters vary. A key tool for studying such questions is the discriminant variety, which partitions the parameter space into regions of constant qualitative and quantitative properties of the solutions. A common challenge is that computing an explicit equation for the discriminant variety requires solving a costly elimination problem. In this talk, I will present a new approach to finding sample points in all connected components of the real complement of a discriminant variety, which uses routing functions and pseudo-witness sets from numerical algebraic geometry to circumvent the need for symbolic elimination. Problems from dynamical systems and the physics of phase diagrams will be used as motivating examples. This is joint work in progress with Paul Breiding, John Cobb, Aviva Englander, Nayda Farnsworth, Jonathan Hauenstein, David Johnson, Jordy Lopez Garcia, and Deepak Mundayur.

- 11.00 - 11.50 (Invited talk): **Lorenzo Baldi** - University of Leipzig (Leipzig).

  Toric extensions of Pólya's theorem

  Abstract: The classical Pólya's theorem provides a simple method for certifying that a homogeneous polynomial of degree d is strictly copositive, that is, it takes only positive values on the nonnegative real orthant. However, this method might fail to detect copositivity of polynomials that are missing certain degree d monomials. In this talk, we present extensions and strong converses to Pólya's theorem for sparse polynomials, using techniques from positive toric geometry. If times permits, we will explore how this method can be used to study the convergence of Feynman integrals in particle physics. This is a joint work with Rainer Sinn, Máté L. Telek, and Julian Weigert.

- 13.20 - 14.10 (Invited talk): **Marta Panizzut** - Arctic University of Norway (Tromsø).

  Positive geometry of del Pezzo surfaces

  Abstract: Positive geometry is a recent branch of mathematical physics which presents exciting connections with real, complex and tropical algebraic geometry. In this talk, we introduce the topic by illustrating several examples, such as the positive geometry of polytopes, hyperplane arrangements and toric varieties. We will then focus on del Pezzo surfaces and their moduli spaces. If time permits, we will also consider the case of toric arrangements.

- 14.10 - 14.35 (Contributed talk): Pending of confirmation

- 15.10 - 15.35 (Contributed talk): **Erin Connelly** - University of Osnabrück (Osnabrück).

  A computer vision problem in flatland

  Abstract: When is it possible to project two sets of labeled points of equal cardinality lying in a pair of projective planes to the same image on a projective line? We give a complete answer to this question, obtaining the following results. We first show that such a pair of projections exist if and only if the two point sets are themselves images of a common point set in projective space. Moreover, we find that for generic pairs of point sets, a common projection exists if and only if their cardinality is at most seven. In these cases, we give an explicit description of the loci of projection centers that enable a common image.

- 15.35 - 16.00 (Contributed talk): **Jiayi Li** - MPI of molecular cell biology and genetics (Dresden).

  Local learning coefficients in transformers: An algebro-geometric lens on generalization in neural networks

  Abstract: Why do heavily over-parameterized neural networks often generalize far better than classical statistical theory predicts, even after they have enough capacity to memorize the training data? This question sits at the heart of modern machine learning theory. Singular Learning Theory (SLT) tackles the puzzle through an algebraic-geometric measure: the local learning coefficient—equivalently, the real log-canonical threshold (RLCT). The RLCT quantifies the parameter-space singularities where identifiability fails and the Fisher information degenerates. It dictates the leading $n-1n^{-1}n-1$ term in the asymptotic expansion of Bayesian free energy and expected generalization error, thereby extending classical information criteria to today's degenerate, high-capacity models, such as neural networks. In this talk, I will present a recent development of SLT for transformer architectures. In particular, we show that local learning coefficients predicts the generalization behavior of neural networks, and can be applied to defect and control the renowned 'grokking' phenomenon.

- 16.00 - 16.25 (Contributed talk): **Jules Tsukahara** - Institut de Mathématiques de Jussieu-Paris Rive Gauche and INRIA (Paris).

  Computing real log canonical thresholds of bivariate polynomials with rational coefficient

  Abstract: In algebraic geometry, the real log canonical threshold (RLCT) is a birational invariant that measures, in some sense, the complexity of the real singularities of a polynomial. Interestingly, RLCTs also appear in machine learning, where they are used to define a generalization of the Bayesian Information Criterion (BIC) to singular statistical models, including, for example, neural networks and hidden Markov models. The BIC of a model measures its complexity and is used for hyperparameter selection. Unfortunately, computing the RLCT of a singularity is a difficult task, as it depends on its resolution of singularities. We present an efficient algorithm to compute the RLCT of bivariate polynomials with rational coefficients, based on the theory of Newton-Puiseux expansions, for which we give precise complexity estimates. We illustrate the algorithm with examples from exponential families and graphical models.

Wednesday 25th of March 2026

- 9.00 - 9.50 (Invited talk): **Beatriz Pascual-Escudero** - Universidad Politécnica de Madrid (Madrid).

  Real and positive solutions of vertically parametrized systems

  Abstract: By vertically parametrized systems we mean polynomial systems where all polynomials arise as linear combinations of a fixed set of monomials, each of which is scaled by a parameter. These systems appear, for example, in the study of steady states of mass action or power-law systems from reaction networks. We will discuss the geometry of the sets of real and positive solutions of such systems, meaning its existence, dimension and nondegeneracy for generic parameter values.

- 9.50 - 10.15 (Contributed talk): **Gökçen Dilaver Tunç** - Bursa Technical University (Bursa).

  Extending generalized splines

  Abstract: Splines, originating in engineering design, have become a central mathematical tool for piecewise polynomial functions with smoothness constraints. They play an important role in approximation theory, numerical analysis, and the numerical solution of partial differential equations. Billera and Rose reinterpreted classical splines via the *dual graph* of a polyhedral complex, leading to the theory of generalized splines, where vertex labels and edge constraints encode spline conditions. This framework naturally arises in algebraic topology and combinatorics, and has been used to study equivariant cohomology rings of toric and algebraic varieties. We present recent progress on extending generalized splines, a framework that broadens this setting by assigning modules to vertices rather than restricting to elements of a fixed ring. We reformulate and extend the GKM-matrix construction and, alongside Gröbner basis methods, develop alternative techniques to construct bases and study structural properties. These results establish a foundation for further study at the interface of algebraic combinatorics, module theory, and computational algebra.

- 10.15 - 10.40 (Contributed talk): **Angelo El Saliby** - MPI for Mathematics in the Sciences (Leipzig).

  Global rigidity of highly regular vertex-transitive graphs

  Abstract: Unique realizability of graphs has long been an important problem, for example in molecular biology. There, measurements of the distances between some parts of a molecule might be known, and one wishes to infer the whole structure of the molecule. Before attempting such a task, it is crucial to verify that the problem is well posed. We do so by understanding whether the combinatorial data, encoded in an undirected graph, forces a locally unique reconstruction or, even better, a globally unique one. The latter property is called global rigidity of graphs. In 2010 it was proved that global rigidity is a generic property, i.e., reconstruction is either almost always globally unique, or almost never globally unique. In 2023, Soma Villànyi showed that high vertex connectivity implies global rigidity. This resolves in particular a conjecture dating back to 1982. This result uses recent advancements in the theory of weakly globally linked pairs. After introducing rigidity from a matroid theoretic point of view and carefully analyzing weak global linkedness and it's use by Soma Villànyi, we apply the same strategy to show that all undirected Cayley graphs of high enough degree are globally rigid. Furthermore, we resolve a conjecture of Sean Dewar, by proving that the same result holds for vertex transitive graphs. In the latter case we also show that our result is tight.

- 11.00 - 11.50 (Invited talk): **Marta Casnellas** - Universitat Politècnica de Catalunya (Barcelona).

  Using algebraic geometry to identify phylogenetic networks

  Abstract: One of the prior steps to perform inference on phylogenetic networks is ensuring that the phylogenetic network can be identified from the distributions generated by a Markov process on it. Algebraic equations that allow distinguishing a phylogenetic network from another have been provided by computational algebra when the Markov process is sufficiently simple. In this talk, we will explain how to obtain equations for networks evolving under equivariant models (which include the previous but also the general Markov model and other submodels) based on rank conditions from flattenings of tensors. These equations can be used to prove that certain networks are distinguishable.