CLAUDIA SCHEIMBAUER, LECTURE 5

RECOLLECTION:

$$\begin{tikzcd} \mathrm{Disk}_n^{fr,\sqcup} \arrow[r, "A"', "\text{sym. mon.}"] \arrow[d, hook] & (\mathcal{S},\otimes) \\ \mathrm{Mfd}_n^{fr,\sqcup} \arrow[ur, dashed, "LKE"] \arrow[r, dashed, "\int A"] & {} \end{tikzcd}$$

$\leftrightarrow$ E_n-ALG IN $\mathcal{S}$

EX: $\underline{n=1}$, $\mathcal{S}=\mathrm{Vect}$ $\to$ ALGEBRAS

$\underline{n=2}$ $\mathcal{S}=\mathrm{Cat}$ $\to$ BRAIDED MON. CATEGORIES

$\underline{n=1}$, $\mathcal{S}=\mathrm{Ch}$ $\rightsquigarrow$ A_∞-ALG

ANY $\underline{n}$, $\mathcal{S}=\mathrm{Top}$ $\rightsquigarrow$ n-FOLD LOOP SPACES

FACTORIZATION HOMOLOGY AKA TOPOLOGICAL CHIRAL HOMOLOGY [LURIE]

=

LEFT KAN EXTENSION.

HAVE AN EXPLICIT FORMULA:

$$\int_M A = \operatorname*{colim}_{\mathrm{Disk}_n^{fr}/M} A$$

LKE EXISTS, IS SYMM. MONOIDAL & AND DESCRIBED THIS WAY.

[AYALA-FRANCIS-(TANAKA)] $\rightsquigarrow$ THIS CATEGORY IS <u>SIFTED</u>

ASSUME $(\mathcal{S},\otimes)$ HAS SIFTED COLIMITS THAT COMMUTE WITH THE $-\otimes-$ IN EACH VARIABLE: "$\otimes$-SIFTED COCOMPLETE"

REM: 1) $\mathrm{Disk}_n^{fr} \simeq \mathrm{Disk}_n^{fr}[(D_1 \hookrightarrow D_2)^{-1}]$

[AF(T)]

Disk_n^{fr} = DISCRETE VERSION, MORPHISMS = SET OF EMBEDDINGS

INVERT EMBEDDINGS OF 1 DISC IN 1 DISC

2) HIGHER HOCHSCHILD HOMOLOGY: FOR E_∞-ALG / COMM. ALGEBRAS / dgas

$$
\begin{tikzcd}
\mathrm{Fin}^{\amalg} \arrow[r, "R"] \arrow[d, hook] & (\mathcal{J}, \otimes) \\
s\mathrm{Set} \arrow[ur, dashed, "\mathrm{LKE}", "{HH(-,R)}"'] &
\end{tikzcd}
$$

3) GENERALIZED HOMOLOGY FOR MFDS ⇝ VARIANT OF E-S AXIOMS.

VARIATIONS:

•) OTHER TANGENTIAL STRUCTURES, eg ORIENTATIONS AS IN THE OTHER LECTURE SERIES.

•) STRATIFIED MANIFOLDS

[AFT] COEFF. ~ E_n-ALG + BIMODULES + MODULES ...

[FRANCIS–ROZENBLYUM] COEFF = (∞, n)-CATEGORIES

•) ADD G-ACTIONS. [MÜLLER–WOIKE]: SPACES WITH PRINCIPLE G-BDLS ⇝ $HH^G(-,-)$

[HOREV] [WEELINCK] FACT. HOMOLOGY WITH (GENUINE) G-ACTION G FINITE.

STANDING ASSUMPTION: $(\mathcal{S}, \otimes)$ $\otimes$-SIFTED COCOMPLETE

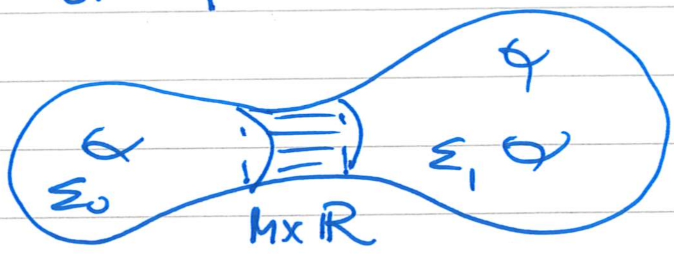
THM (EXCISION) FOR FRAMED n-MFD Σ, DECOMPOSITION

$$\Sigma = \Sigma_0 \cup_{M\times\mathbb{R}} \Sigma_1 , \quad \Sigma_0, \Sigma_1 \text{ OPEN IN } \Sigma$$

$$\Sigma_0 \cap \Sigma_1 \cong M \times \mathbb{R}$$

THEN

$$\int_\Sigma A \xleftarrow{\simeq} \int_{\Sigma_0} A \overset{(\mathbb{L})}{\underset{\int_{M\times\mathbb{R}} A}{\otimes}} \int_{\Sigma_1} A$$



WHY DOES THIS MAKE SENSE?

$$\mathbb{R} \amalg \mathbb{R} \hookrightarrow \mathbb{R}$$

$$\rightsquigarrow (M\times\mathbb{R}) \amalg (M\times\mathbb{R}) \hookrightarrow (M\times\mathbb{R})$$

$$\rightsquigarrow \int_{M\times\mathbb{R}} A \otimes \int_{M\times\mathbb{R}} A \longrightarrow \int_{M\times\mathbb{R}} A \quad \rightsquigarrow \quad \mathrm{Disk}_1^{fr} \xrightarrow{M\times(-)} \mathrm{Mfd}_n^{fr} \xrightarrow{\int A} \mathcal{S}$$

SIMILARLY, $\int_{M\times\mathbb{R}} A \circlearrowright \int_{\Sigma_i} A$

COBORDISM CATEGORY

$$\Rightarrow \dots \text{"}n\mathrm{Cob}^{\amalg}\text{"} \xrightarrow{\int A} \mathrm{Alg}(\mathcal{S})$$

e.g. USING $\mathcal{S}$ = CHAIN CPXS
i.e. OBJ dga's
MORPH BIMODULES IN Ch

$$\left[\begin{array}{l} \text{OBJ } M \longmapsto \int_{M\times\mathbb{R}} A \\ \text{MORPH } \Sigma \longmapsto \int_\Sigma A \end{array}\right.$$

GOAL: FULLY EXTEND THIS THEOREM!

- $n\mathrm{Cob} \rightsquigarrow \mathrm{Bord}_n$ [HOREL $(\infty,1)$], [(∞,n) S PHD THESIS]
- REPLACE TARGET.

FACTORIZATION ALGEBRAS

DEF M n-MFD. THE DISCRETE COLORED OPERAD $\mathrm{Disk}(M)$ HAS COLORS GIVEN BY OPEN SUBSETS OF M DIFFEOMORPHIC TO $\amalg_k \mathbb{R}^n$, AND FOR $U_1, \dots, U_n, V$ COLORS,

$$\mathrm{Disk}(M)(U_1, \dots, U_n; V) := \begin{cases} * & \text{IF } U_1 \amalg \dots \amalg U_n \subseteq V \\ \emptyset & \text{OTHERWISE} \end{cases}$$

A PREFACTORIZATION ALGEBRA ON M IN $\mathcal{S}$ IS A $\mathrm{Disk}(M)$-ALGEBRA IN $\mathcal{S}$. A FACTORIZATION ALGEBRA ON M VALUED IN $\mathcal{S}$ IS A PREFACTORIZATION ALG $\mathcal{F}$ S.T.

(1) $\underbrace{\mathcal{F}(U_1) \otimes \dots \otimes \mathcal{F}(U_n)}_{U_i\text{'s DISJOINT}} \xrightarrow[\text{STRUCT MAP}]{\simeq} \mathcal{F}(\underbrace{U_1 \amalg \dots \amalg U_n}_{\overset{\shortparallel}{V}})$

(2) codescent FOR WEISS COVERS.

EXAMPLE: BY $\mathrm{Disk}_n^{fr} \simeq \mathrm{Disk}_n^{fr}[(D_1 \hookrightarrow D_2)^{-1}]$ (OR FOR ANOTHER VERSION, SEE [GINOT-TRADLER-ZEINALIAN])

GIVEN A E_n-ALG, M FRAMED MFD, $U \mapsto \int_U A$ IS A LOCALLY CONSTANT FACT. ALG ON M.

REM: LURIE $(M = \mathbb{R}^n)$ $\mathrm{Disk}_n^{fr}[(D_1 \hookrightarrow D_2)^{-1}] \longrightarrow E_n$

$\rightsquigarrow E_n\text{-ALG} \xrightarrow{\simeq} \mathrm{Fact}^{l.c.}(\mathbb{R}^n)$

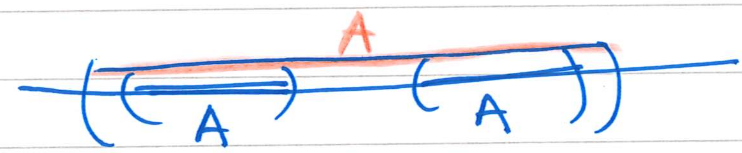
DEF: A LOCALLY CONSTANT FACT. ALG ON M IS A PREFACT. ALG $\mathcal{F}$ S.T. $\mathcal{F}(D_1) \xrightarrow{\simeq} \mathcal{F}(D_2)$ $\forall D_1 \hookrightarrow D_2$ DISCS INCLUSIONS

[COSTELLO–GWILLIAM] ⇝ LOCALLY CONSTANT CONDITION RELATED TO TOPOLOGICAL QUANTUM FIELD THEORIES...

<u>EX</u>: A ALGEBRA ⇝ LOCALLY CONSTANT FACTORIZATION ALG ON $\mathbb{R}$:

ANY INTERVAL $\longmapsto A$

$\sqcup \longmapsto \otimes$



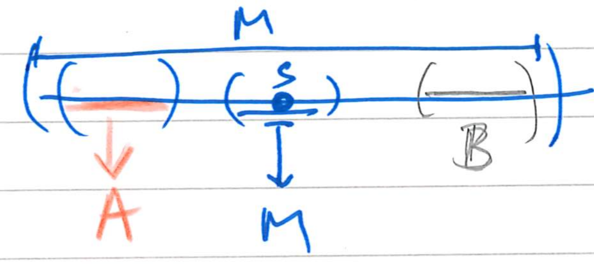
⇝ $A \otimes A \longrightarrow A$

MULTIPLICATION.

CHECK ASSOCIATIVITY. <u>NOTE</u>: LOCALLY CONSTANT.

<u>EX</u>: ${}_A\circlearrowright M \circlearrowleft_B$ ⇝ FACTORIZATION ALG ON $(\mathbb{R}, s)$

BIMODULE



⇝ $A \otimes M \otimes B \longrightarrow M$

<u>NOTE</u>: NOT LOCALLY CONSTANT.