Claudia Scheimbauer, Lecture 4

Yesterday: A Segal space is a simplicial space $X : \Delta^{op} \to \text{Spaces}$ s.t.

$$\forall n \geq 1, \quad X_n \xrightarrow{\simeq} X_1 \overset{h}{\underset{X_0}{\times}} \cdots \overset{h}{\underset{X_0}{\times}} X_1$$

Think of X_0 = objects, X_1 = morphisms

X_2 = pairs of composable morphisms with a choice of composition.

Note: X_1 is a space, so 2-morphisms are paths in X_1.

Bordisms as a Segal space

Rough idea: $N_\cdot(n\mathrm{Cob})$ is a simplicial space and it is a Segal space.

We will look at a variation of this definition, more suitable for later purposes.

$$X = \mathrm{Bord}_n^{(\infty,1)}$$

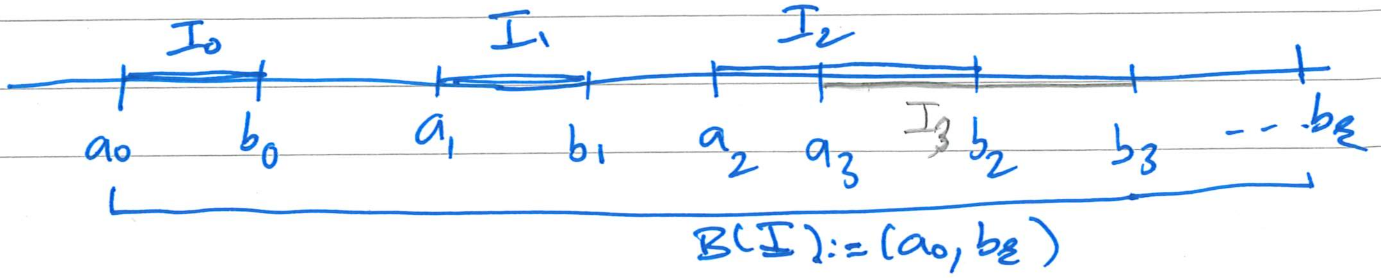
X_ℓ = space of ... $(M, \underline{I} = (I_0 \leq \cdots \leq I_\ell))$

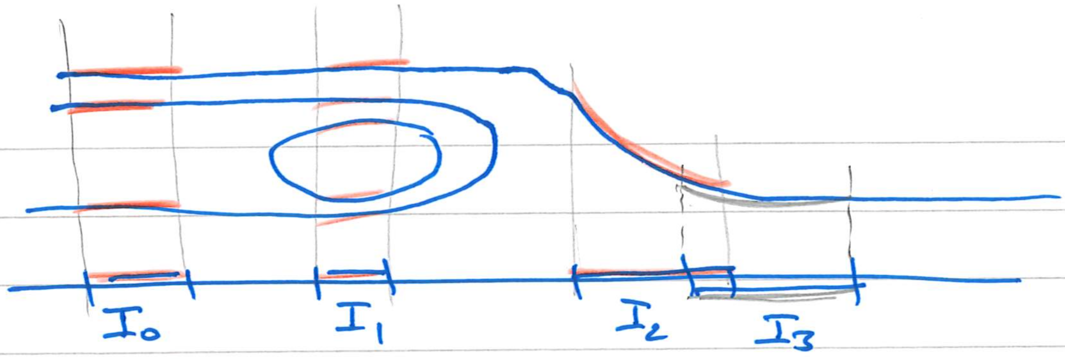
"intervals" $[a_i, b_i]$

$M \subseteq B(\underline{I}) \times \mathbb{R}^\infty$

$a_0 \leq \cdots \leq a_\ell, \; b_0 \leq \cdots \leq b_\ell$

$a_i < b_i$





S.t. $M \subseteq B(\underline{I}) \times \mathbb{R}^\infty$

$$\begin{tikzcd} M \arrow[r, phantom, "\subseteq"] \arrow[dr, "\pi"'] & B(\underline{I}) \times \mathbb{R}^\infty \arrow[d, "pr"] \\ & B(\underline{I}) \end{tikzcd}$$

(1) π is PROPER

(2) OVER $_{B(\underline{I})}$ $([a_i, b_i] = I_i)$, π is SUBMERSIVE

REPLACES CYLINDRICITY.

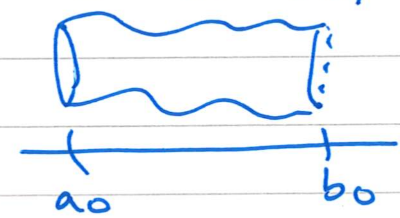
IN PARTICULAR:

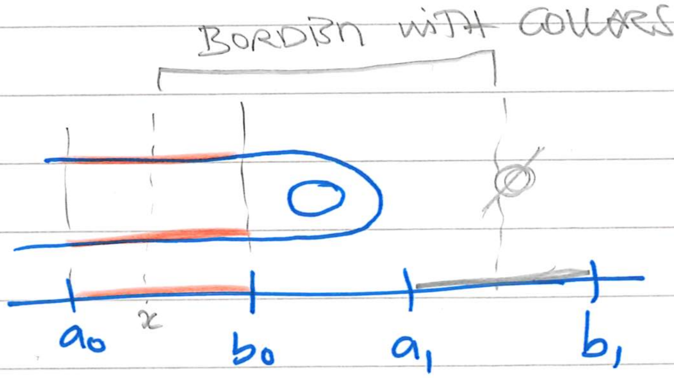
$X_0 \ni (M, I_0)$ $\qquad B(I_0) = (a_0, b_0)$, $I_0 = [a_0, b_0]$

$\pi^{-1}(B(I_0)) \cong$ CYLINDER

$\pi^{-1}(x) \times (a_0, b_0)$

FOR ANY $x \in (a_0, b_0)$



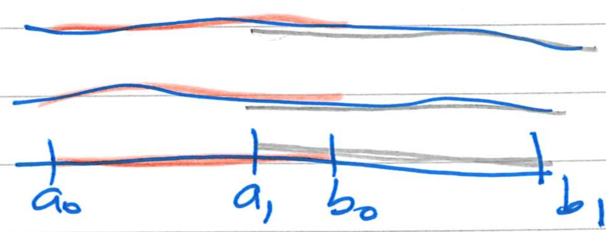


$X_1 \ni (M, (I_0, I_1))$

IF THE INTERVALS OVERLAP:

LONG CYLINDER

$\rightsquigarrow$ will ALLOW TO DEFINE ~~DEGENERACIES~~ = DOUBLING AN INTERVAL



FACE MAPS: DELETING INTERVALS.

REFS: LURIE, CALAQUE-S.

NOTE: FOR THE PURPOSE OF HAVING A SEGAL SPACE, WE COULD HAVE USED POINTS INSTEAD OF THE INTERVALS I_i, BUT INTERVALS WILL BE USEFUL LATER FOR BUILDING TFT'S.

NOW, (∞,n)-VERSION.

DEF: AN (∞,n)-CATEGORY IS (FOR US) A (COMPLETE) n-FOLD SEGAL SPACE, i.e. AN n-FOLD SIMPLICIAL SPACE

$$X : (\Delta^{op})^n \longrightarrow \text{SPACES}$$

s.t. (1) $X_{k_1,\dots,k_{i-1},\bullet,k_{i+1},\dots,k_n} : \Delta^{op} \longrightarrow \text{SPACES}$

is (COMPLETE) SEGAL

(2) $X_{k_1,\dots,k_n,0,0,\dots,\bullet}$ is EQUIVALENT ~~to~~

(i.e. LEVEL-WISE EQUIV.) TO A CONSTANT $(n-i-1)$-FOLD SEGAL SPACE, i.e.

$$X_{k_1,\dots,k_{i-1},0,\dots,0} \xrightarrow[\simeq]{deg} X_{k_1,\dots,k_{i-1},0,k_{i+1},\dots,k_n}$$

$n=2$: (1) $X_{k,\bullet}$ AND $X_{\bullet,k}$ ARE (COMPLETE) SEGAL SPACES

(2) $X_{0,\bullet}$ is ESSENTIALLY CONSTANT:

$$X_{0,0} \xrightarrow{\simeq} X_{0,k}$$

OBJECTS: $X_{0,0}$, 1-MORPH $= X_{1,0}$, 2-MORPH $= X_{1,1}$

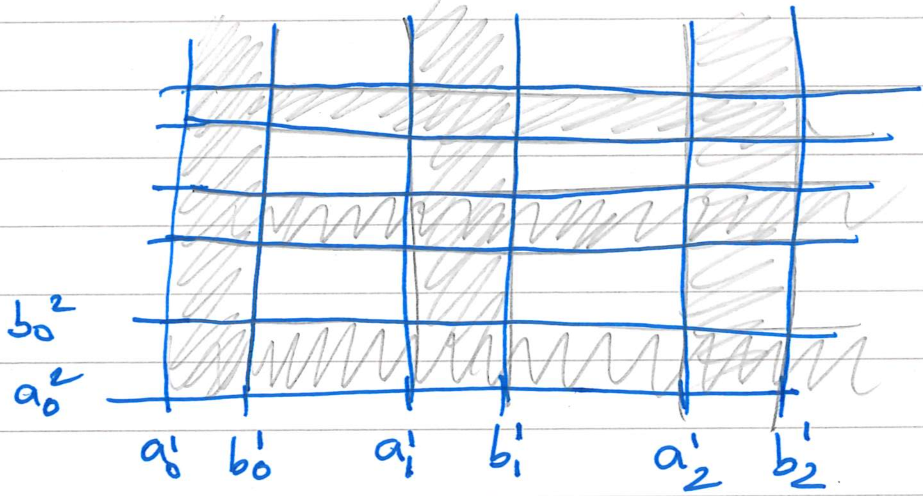
$$\begin{tikzcd} X_{0,0} \arrow[d, "\simeq"] \\ X_{0,1} \end{tikzcd}$$ BECAUSE $\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & \bullet \end{tikzcd}$ INSTEAD OF $\begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] \arrow[ur, phantom, "\Nearrow"] & \bullet \arrow[u] \end{tikzcd}$

NOTE: NOW WE HAVE 2-MORPHISMS ~~[illegible]~~ AS PETALS IN $X_{1,0}$ AND AS POINTS IN $X_{1,1}$.
COMPLETENESS: $X_{1,0}$ ~~[illegible]~~ is EQUIVALENT TO THE GROUPOID OF INVERTIBLE 2-MORPH FROM $X_{1,1}$.

BORDISMS $X_{k_1,\dots,k_n}$ FROM $X_{\bullet,\dots,\bullet} = \mathrm{Bord}^{(\infty,n)}_{n\;\bullet,\dots,\bullet}$

$$\begin{gathered} \ni \\ \left(M, (\underline{I}^i)_{1\leq\dots\leq n}\right) \end{gathered}$$

n=2



$\subseteq \mathbb{R}^n$

$\downarrow$

$\mathbb{R}^s$

$$B(\underline{I}^i) := (a_0^1, b_{k_1}^1) \times \dots \times (a_0^n, b_{k_n}^n)$$

$$\begin{tikzcd} M \subset B(\underline{I}^i) \times \mathbb{R}^\infty \arrow[d, "pr"] \\ B(\underline{I}^i) \end{tikzcd}$$

$\pi : M \to B(\underline{I}^i)$

s.t. (1) π is PROPER

(2) $p_s : M \xrightarrow{\pi} B(\underline{I}^i) \to \mathbb{R}^s$, $s \subset \{1,\dots,n\}$, AT $x \in p_{\{i\}}^{-1}(I_j^i)$, THE $p_{\{i,\dots,n\}}$ IS SUBMERSIVE

WARNING: NOT COMPLETE!

FACTORIZATION HOMOLOGY ... TO BE DIGESTED

DEFINITION: THE TOPOLOGICALLY ENRICHED ($\rightsquigarrow$ $(\infty,1)$-CAT) OF FRAMED n-MANIFOLDS Mfd_n^{fr} HAS

OBJECTS = SMOOTH MANIFOLDS WITH FRAMING

$$\begin{tikzcd} & \{*\} \arrow[d, "p"] \\ M \arrow[ur, dashed, "\phi"] \arrow[r, "\tau_M"'] & BGL_n \end{tikzcd}$$

$(\Longleftrightarrow TM \simeq M \times \mathbb{R}^n)$

THERE IS A HTPY $p \circ \phi \simeq \tau_M$

MORPHISMS = SPACE OF EMBEDDINGS COMPATIBLE WITH FRAMING:

$$\begin{tikzcd} Mfd_n^{fr} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Top/_* \arrow[d] \\ Mfd_n \arrow[r] & Top/_{BGL_n} \end{tikzcd}$$

$$M \longmapsto \tau_M$$

is A PULL-BACK OF ∞-CAT.

$Mfd_n^{fr} \supset \mathcal{D}iscs_n^{fr}$ SYMM. MONOIDAL WITH $\amalg$

OBJECTS = FINITE DISJOINT UNIONS OF FRAMED DISCS $\simeq \coprod_k \mathbb{R}^n$

IDEA:

DEF / PROP: AN E_n-ALGEBRA IN A SYM. MONOIDAL $(\infty,1)$-CATEGORY $\mathcal{S}$ IS A SYMM. MONOIDAL FUNCTOR $\mathcal{A} : \mathcal{D}isc_n^{fr} \longrightarrow \mathcal{S}$

EXERCISE: $n=1$, $\mathcal{S} = \mathrm{Vect}$, THIS IS JUST AN ASSOCIATIVE ALGEBRA!

__Def:__ Given A as above, __factorisation homology__ is the left Kan extension (of sym. monoidal $(\infty,1)$-categories)

$$\begin{tikzcd} \mathcal{D}isk_n^{fr} \arrow[r, "A"] \arrow[d, hook] & \mathcal{S} \\ \mathcal{M}fd_n^{fr} \arrow[ur, dashed, "\underline{\int A}"'] & \end{tikzcd}$$

For appropriate $\mathcal{S}^{\otimes}$ (= tensor sifted cocomplete), it is the usual left Kan extension.