Claudia Scheimbauer, Lecture 4

Yesterday: A Segal space is a simplicial space $X : \Delta^{op} \to \text{Spaces}$ s.t.
\[
\forall n \geq 1, \quad X_n \xrightarrow{\sim} X_1 \times \cdots \times X_1
\]

Think of $X_0 = \text{objects}$, $X_1 = \text{morphisms}$

$X_2 = \text{pair of composable morphisms with a choice of composition}.$

Note: $X_1$ is a space, so 2-morphisms are paths in $X_1$.

Bordisms as a Segal space

Rough idea: $N(n\text{-Cob})$ is a simplicial space and it is a Segal space.

We will look at a variation of this definition, more suitable for later purpose.

$X = \text{Bord}_n$

$X_2 = \text{space of} \quad (M, I = (I_0 \leq \cdots \leq I_2))$

$M \leq B(I) \times \mathbb{R}^\infty$

$B(I) = (a_0, b_2)$

\[
\begin{array}{cccccccc}
& & I_1 & & I_2 & & & \\
& a_0 & b_0 & a_1 & b_1 & a_2 & a_3 & b_2 & b_3 & \cdots & b_2 \\
\end{array}
\]
S.t. \( \mathbf{M} = B(I) \times \mathbb{R}^n \)  
(1) \( \mathbf{M} \) is proper 
(2) \( \mathbf{M} \) over \( B(I) \)  
\( [a_i,b_i] = I_i \)  
\( \tau \) is submersive 

\[ \begin{align*} 
\text{In particular:} & \\
X_0 & \cong (\mathbf{M}, I_0) \\
\pi^{-1}(B(I)) & \cong \text{cylinder} \\
\pi^{-1}(x) & \times (a_0, b_0) \\
\text{for any } x \in (a_0, b_0) \\
\end{align*} \]

\[ \begin{align*} 
X_1 & \cong (\mathbf{M}, (I_0, I_1)) \\
\end{align*} \]

If the intervals overlap:
Long cylinder

\( n \rightarrow \) will allow to 
Define new generator:
Doubling an interval
Face maps: redefining interval.

Refs: Lurie, Cattaneo-S.
NOTE: FOR THE PURPOSE OF HAVING A SEGAL
SPACE, WE COULD HAVE USED POINTS
INSTANCE OF THE INTERVALS I_i; BUT
INTERVALS WILL BE USEFUL LATER
FOR BUILDING TH' S.

Now, (dgn)-version.

**Def:** An \((\infty, n)\)-category is (for us) a
(complete) \(n\)-fold Segal space, i.e.
an \(n\)-fold simplicial space

\[ X : (\Delta^{op})^n \to \text{Spaces} \]

s.t. (1) \(X_{k_1, \ldots, k_n} : \Delta^0 \to \text{Spaces} \)

is (complete) Segal

(2) \(X_{k_1, \ldots, k_n, 0, 0, \ldots} \) is equivalent to

(i.e. level-wise equiv.) to a constant

\((n-i-1)\)-fold Segal space, i.e.

\[ X_{k_1, \ldots, k_{i+1}, 0, \ldots} \xrightarrow{\text{deg}} X_{k_1, \ldots, k_{i+1}, 0, k_{i+2}, \ldots, k_n} \]

\(n=2\): (1) \(X_{k_1, 0}\) and \(X_{0, k_2}\) are (complete) Segal space

(2) \(X_{0, 0}\) is essentially constant:

\[ X_{0, 0} \rightleftarrows X_{0, 1} \]

OBJECTS: \(X_{0, 0}\) \(1\)-MORPH \(= X_{1, 0}\), \(2\)-MORPH \(= X_{1, 1}\)

\[ X_{0, 1} \xrightarrow{\text{because}} \]

\[ \uparrow \quad \text{instead of} \quad \Uparrow \]
Note: Now we have 2-morphisms as paths in $X_{1,0}$ and as points in $X_{1,1}$.

Completeness: $X_{1,0}$ is equivalent to the groupoid of invertible 2-morphisms from $X_{1,1}$.

Borders $X_{a_0, \ldots, b_n}$ from $X_{0, \ldots, 0} = \text{Bord}_{n, 0, 0}$

\[
(M, (I^i), 1 \leq i \leq n)
\]

$n = 2$

\[
\begin{array}{cccc}
  & b_0 & & b_2 \\
  a_0 & \cdots & a_1 & \cdots & a_2 \\
  & b_1 & & b_2
\end{array}
\]

$B(I^i) := (a_0^i, b_0^i) \times \cdots \times (a_n^i, b_n^i)$

$M \subset B(I^i) \times \mathbb{R}^\infty$ s.t.

1. It is proper

2. $\pi: M^{\text{op}} \to B(I^i) \to \mathbb{R}^S$

$s \subset \{1, \ldots, n\}$, at $x \in \text{Pr}^s(I^i)$, the $\text{Pr}_1 \ldots \text{Pr}_s$ is surjective

Warning: Not complete!
**Factorization Homology -- To Be Digested**

**Definition:** The topologically enriched (∞, 1)-category of framed n-manifolds \( \text{Mfd}^f_n \) has

**Objects:** Smooth manifolds with framing

\[
\begin{align*}
\phi & \colon \mathbb{R}^n \\
\downarrow & \\
M & \to \text{BGL}_n
\end{align*}
\]

\( \iff \text{TM} \cong M \times \mathbb{R}^n \)

There is a HFP \( p \circ \phi \cong \pi_n \)

**Morphisms:** Space of embeddings compatible with framing:

\[
\begin{align*}
\text{Mfd}^f_n & \to \text{Top}^f/ \\
\downarrow & \\
\text{Mfd}^f_n & \to \text{Top}^f/\text{BGL}_n
\end{align*}
\]

\( \iff \) is a pull-back of \( \omega \)-cat.

\[
\begin{align*}
\text{Mfd}^f_n \cong \text{Disc}_n \\
\text{synth. monoidal with } \mathbb{I}
\end{align*}
\]

**Objects:** Finite disjoint unions of framed discs

\[
\begin{align*}
\cong \mathbb{I} \times \mathbb{R}^n
\end{align*}
\]

**Def/Prop:** An \( E_n \)-algebra in a synth. monoidal \((\infty, 1)\)-category \( \mathcal{C} \) is a synth. monoidal functor \( A : \text{Disc}^f_n \to \mathcal{C} \)

**Exercise:** \( n = 1 \), \( \mathcal{C} = \text{Vect} \), this is just an associative algebra!
Def: Given A as above, factorisation homology is the left Kan extension 
(of sym. monoidal (59)-categories).

For appropriate $f \colon \text{Disk}_n \to \mathcal{Y}$, it is the usual left Kan extension.