

Claudia Scheimbauer, Lecture 3

(Crash course on \((\infty,1)\)-categories)

CONTINUED

From yesterday:

\textbf{Def:} A \((\infty,1)\)-category is a category enriched in spaces.

\textit{\(\Rightarrow\)}

Either \(\text{Top} = \text{compactly generated Hausdorff spaces}\)

\textit{\(\Rightarrow\)}

\(\text{Sing} \downarrow \text{N}.\)

\(\text{or } \text{sSet} = \text{simplicial sets}\)

\((\text{equivalent categories})\)

\(\textbf{Ex.}\) 1 Top or sSet

2 Any simplicial model category? "Presentation of \((\infty,1)\)-Cat"?

"Presentable \((\infty,1)\)-categories are presentable in this sense..."

3 Coherently \textit{add} category.

Recall/learn: Whitney's Embedding Theorem:

Any smooth n-dimensional manifold can be smoothly embedded in \(\mathbb{R}^{n+1} \times \mathbb{R}^\infty\) with \(\mathbb{R}^\infty\) corners.
• CAN BE MODIFIED FOR BORDCONS \( \sim [0,1] \times \mathbb{R}^\infty \)
  or BORDCONS OF BORDCONS
  \( \sim [0,1]^2 \times \mathbb{R}^\infty \)
  (REFS: [Laueres], [Calaque-S])

• SPACE OF SUCH EMBEDDINGS IS CONTRACTIBLE.

FIRST VERSION OF (3): \( \mathcal{C} \text{ob} \) (THUAN, MADSEN)
[GMTW]

OBJECTS: Pairs \((M, a)\) as \(\mathbb{R}^\infty\)
\(M \text{ closed } (n-1)-\text{dim submanifold of } \mathbb{R}^\infty\).

MORPHISMS: NON-IDENTITY FROM \((M_0, a_0)\) TO \((M, a)\) IS
A TRAPEZE \((W, a_0, a_1)\) \(a_0 < a_1\), AND
\(W \text{ compact submanifold } W \subseteq [a_0, a_1] \times \mathbb{R}^\infty\)
\(W \mid_{a_0 \times \mathbb{R}^\infty} = M_0 + \text{coarses}\):

1. \(\exists W = W \cap ([a_0, a_1] \times \mathbb{R}^\infty)\)
2. \(\exists \varepsilon > 0 \text{ s.t. } W \cap ([a_1 - \varepsilon, a_1] \times \mathbb{R}^\infty) = (a_1 - \varepsilon, a_1] \times M_1\)
3. \(W \cap ([a_0, a_0 + \varepsilon] \times \mathbb{R}^\infty) = [a_0, a_0 + \varepsilon] \times M_0\)

\(\Rightarrow \text{ CAN COMPOSE DIRECTLY}.

TOPOLOGY: \(\text{Emb}(M, \mathbb{R}^\infty) = \text{Coli} \text{Emb}(M, \mathbb{R}^\infty) \sim \mathbb{C}^0\)

WHITNEY TOPOLOGY
\[ \text{Emb}(M, T^{\infty}) \rightarrow \text{Emb}(N, T^{\infty}) := \frac{B^{n}(M)}{\text{Diff}(M)} \]

is a principal \( \text{Diff}(M) \)-bundle

Use to topologize:

\[ \text{Obj of } n\text{Cob} \cong \text{TR} \times L \text{L } B^{n}(M) \]

\[ \text{M Diffed class} \]

\[ \text{Morph } \cong \text{ob}(n\text{Cob})_{\text{Lip}_0 < \epsilon, \text{Lip}_1 > 0} \times L \text{L } B^{n}(W) \]

\[ \text{W Diffed class of abstract bordism} (W, \mathcal{O}_0, \mathcal{O}_1, \text{Lip}) \]

\[ \text{Emb}(W, E \text{Lip}) \]

\[ \text{Diff} \]

\[ n\text{Cob. is a category internal to spaces.} \]

[Here] such categories are also \((\infty, 1)\)-categories.

Could also forget the topology of the objects, which would be an equivalent category (although a little strange from some points of view).

Or at least equivalent classifying space

**Note:** Problems with composition fixed by choosing cylindrical collars.

**Second Def:**

**Def:** An \((\infty, 1)\)-category is a quasi-category (= weak Kan complex = oo-category (Lurie)).

That is a simplicial set \( X \) with certain horn lifting properties for "inner horns".
**Lowest Condition**

\[ \Lambda^2 \rightarrow \Delta^2 \]

\[ \Lambda_1 \rightarrow \Delta \]

\[ X_2 \xrightarrow{(d_i^0, d_i^1)} X_1 \times X_1 \]

\[ \text{Fact: The fibers of this map are connectible for an appropriate topology on } X_2 \text{ space.} \]

More generally, need lifts for

\[ \Lambda^n \rightarrow \Delta^n \]

\[ \bigcup_{i \neq k} \Delta^n \rightarrow X \]

**Example:**

1) Nerve of a category
2) \( \text{Sing}(X) \) (satisfies lifting for all horns!)

**Remark:** Great for category theory: (60) lim, Kan ext, over-cats, Quillen this and B, ...

(Joyal, Lurie Higher Topos Theory)
**Third Definition**

**Def:** An \((\infty,1)\)-category \(\mathbf{C}\) is a complete Segal space, that is a simplicial space \(X : \Delta^{op} \to \text{Spaces}\) such that:

1. \((\text{Segal})\) for \(n \geq 1\), \(X_n \xrightarrow{w.e.} \prod_{i=0}^{n-1} X_i x X \to X_0 x X \to X_0\) is induced from \(0 < i \to i+1 - i \in [n]^{[i]} [n]\).

2. \((\text{Completeness})\) \(X_0\) encodes the underlying \(\pi_0\)-groupoid.

Recall \(I[1] = \{0 \leftarrow 1\}\).

Completeness can be reformulated as:

\[ X_0 = \text{Map}^h([0], X_0) \cong \text{Map}^h(N(I[1]), X_0). \]

**Examples:**

1. If \(\mathbf{C}\) is an ordinary category, then \(\mathbf{C}\) satisfies the Segal condition, and if in \(\mathbf{C}\) every isomorphism is an identity, \(\mathbf{C}\) is complete.

2. Modify to \(\mathbf{N}(\mathbf{C}, \text{Iso}\mathbf{C})\) always a complete Segal space.

3. Given "relative category" \(\mathbf{C} \xrightarrow{\text{weak equiv}} \mathbf{N}(\mathbf{C}, \mathbf{W})\) also complete Segal space.

\([\text{Rezk, Barwick-Kan, Mazel-Gee}]\)
Bond\(_n\) \text{ (write) is NOT COMPLETE.}
(SEE TOMORROW)

(Infinity, Toën \((\infty,1)\), Barwick-Scheimbauer-Prieß\((\infty,1)\))

Up to an action of \(\frac{1}{2}2^n\), an models of \((\infty,n)\)-categories are equivalent.

Often have a model category whose fibrant objects are the given \((\infty,1)\)-Cat, or an \((\infty,1)\)-Cat of such...

Note: Direct equivalence in practice are not easy/simple.

(Right-Exact) "Model Free" theory of \((\infty,1)\)-categories
\((\infty,n)\)?