CLAUDIA SCHEIMBAUER, LECTURE 3

(CRASH COURSE ON $(\infty,1)$-CATEGORIES CONTINUED)

FROM YESTERDAY:

DEF AN $(\infty,1)$-CATEGORY IS A CATEGORY ENRICHED IN SPACES.

EITHER Top = COMPACTLY GENERATED HAUSDORFF SPACES

$\mathrm{Sing} \uparrow \downarrow |\cdot|$

OR sSets = SIMPLICIAL SETS

(EQUIVALENT CATEGORIES)

EX: ① Top OR sSets

② ANY SIMPLICIAL MODEL CATEGORY COMBINATORIAL $\Big\}$ = "PRESENTATION OF AN $(\infty,1)$-CAT"

("PRESENTABLE $(\infty,1)$-CATEGORIES" ARE PRESENTABLE IN THIS SENSE ...)

③ COBORDISM CATEGORY.

RECALL / LEARN: WHITNEY'S EMBEDDING THEOREM:

ANY SMOOTH n-dim MFD (WITH BDRY / CORNERS) CAN BE SMOOTHLY EMBEDDED IN $\mathbb{R}^\infty = \operatorname{colim}_{k\to\infty} \mathbb{R}^k$ / $\mathbb{R}_{\geq 0} \times \mathbb{R}^\infty$ / $(\mathbb{R}_{\geq 0})^k \times \mathbb{R}^\infty$

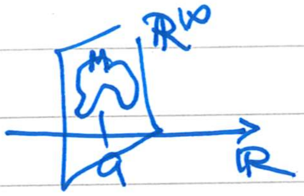
- CAN BE MODIFIED FOR BORDISMS $\rightsquigarrow [0,1]\times\mathbb{R}^\infty$
  OR BORDISMS OF BORDISMS
  $\rightsquigarrow [0,1]^\ell \times \mathbb{R}^\infty$

(REFS: [LAURES], [CALAQUE-S])

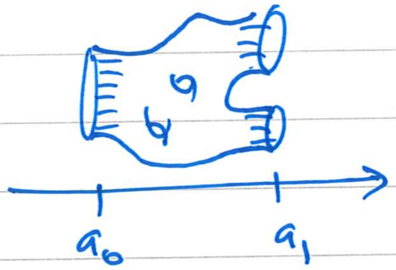
- SPACE OF SUCH EMBEDDINGS IS CONTRACTIBLE.

FIRST VERSION OF 3': $n\mathcal{C}ob$ (TILLMANN, MADSEN [GMTW])

OBJECTS: PAIRS (M,a) $a\in\mathbb{R}$
M CLOSED $(n-1)$-di SUBMFD OF $\mathbb{R}^\infty$.



MORPHISMS: NON-IDENTITY FROM (M_0,a_0) TO (M_1,a_1) IS A TRIPLE (W,a_0,a_1), $a_0<a_1$ AND W COMPACT SUBMFD $W\subseteq[a_0,a_1]\times\mathbb{R}^\infty$ WITH BDRY $W|_{a_i\times\mathbb{R}^\infty}=M_i$ + COLLARS:



(1) $\partial W = W\cap(\{a_0,a_1\}\times\mathbb{R}^\infty)$

(2) $\exists\, \varepsilon>0$ s.t. $W\cap((a_1-\varepsilon,a_1]\times\mathbb{R}^\infty) = (a_1-\varepsilon,a_1]\times M_1$

(3) $W\cap([a_0,a_0+\varepsilon)\times\mathbb{R}^\infty) = [a_0 \times a_0+\varepsilon)\times M_0$

$\rightsquigarrow$ CAN COMPOSE DIRECTLY.

TOPOLOGY: $\mathrm{Emb}(M,\mathbb{R}^\infty) = \mathrm{colim}_{k\to\infty}\, \mathrm{Emb}(M,\mathbb{R}^k)$

WHITNEY C^∞-TOPOLOGY

$$\mathrm{Emb}(M, \mathbb{R}^\infty) \longrightarrow \mathrm{Emb}(M,\mathbb{R}^\infty)/\mathrm{Diff}(M) =: B_\infty(M) \simeq B\mathrm{Diff}(M)$$

IS A PRINCIPLE $\mathrm{Diff}(M)$-BUNDLE

USE TO TOPOLOGIZE:

$$\text{OBJ OF } n\mathcal{C}ob \cong \mathbb{R} \times \coprod_{M \text{ DIFFEO CLASS}} B_\infty(M)$$

$$\text{MORPH} \cong \underbrace{\mathrm{ob}(n\mathcal{C}ob)}_{\text{IDENTITY MORPHISMS}} \amalg \{a_0 < a_1\} \times \coprod_{\substack{W \\ \text{DIFFEO CLASS OF ABSTRACT BORDISM } (W, \partial_0, \partial_1, p)}} B_\infty(W)$$

RESPECTING COLLARS

$\mathrm{Emb}^\partial(W, [0,1] \times \mathbb{R}^\infty) / \mathrm{Diff}$

MAYBE NOT SO NATURAL A DEFINITION

$n\mathcal{C}ob$ IS A CATEGORY <u>INTERNAL TO SPACES</u>.

[HOREL] SUCH CATEGORIES ARE ALSO $(\infty,1)$-CATEGORIES

COULD ALSO FORGET THE TOPOLOGY OF THE OBJECTS, WHICH WOULD BE AN EQUIVALENT CATEGORY (ALTHOUGH A LITTLE STRANGE FROM SOME POINTS OF VIEW).

OR AT LEAST EQUIVALENT CLASSIFYING SPACES

<u>NOTE</u>: PROBLEMS WITH COMPOSITIONS FIXED BY CHOOSING CYLINDRICAL COLLARS.

SECOND ~~NEXT~~ DEF:

<u>DEF</u>: AN <u>$(\infty,1)$-CATEGORY</u> IS A QUASI-CATEGORY (= WEAK KAN COMPLEX = ∞-CATEGORY (LURIE)) THAT IS A SIMPLICIAL SET $X_\bullet$ WITH CERTAIN HORN LIFTING PROPERTIES FOR "INNER HORNS":

LOWEST CONDITION
$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd} \hookrightarrow \blacktriangle$$

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, hook] \arrow[dr] & \Delta^2 \arrow[d, dashed, "\exists \text{ LIFT (NOT UNIQUE)}"] \\ & X_\bullet \end{tikzcd}$$

$$X_2 \xrightarrow[\substack{\shortparallel \\ \text{RESTRICTION} \\ \text{TO } \Lambda^2_1}]{(d^0, d^1)} X_1 \underset{X_0}{\times} X_1$$

(FACT: THE FIBERS OF THIS MAP ARE CONTRACTIBLE FOR AN APPROPRIATE ~~TOPOLOGY ON X_2~~ SPACE...)

MORE GENERALLY, NEED LIFTS FOR

$$\begin{tikzcd} \Lambda^n_k \arrow[r, hook] \arrow[dr] & \Delta^n \arrow[d, dashed, "\exists"] \\ & X_\bullet \end{tikzcd} \qquad \text{FOR ALL } k \neq 0, n.$$

$$\Lambda^n_k = \bigcup_{i \neq k} \delta_i(\Delta^n)$$

EXAMPLES 1) NERVE OF A CATEGORY
2) $Sing(X)$ (SATISFIES LIFTING FOR ALL HORNS!)
$\shortparallel$
KAN COMPLEX

REM: GREAT FOR CATEGORY THEORY: (CO)LIM, KAN EXT, OVER-CATS, QUILLEN THMS A & B, ...
[JOYAL, LURIE HIGHER TOPOS THEORY]

THIRD DEFINITION: [REZK]

DEF: An $(\infty,1)$-category is a complete Segal space, that is a simplicial space

$$X_\bullet : \Delta^{op} \longrightarrow \underset{\text{Kan complexes}}{\overset{\shortparallel}{\text{Spaces}}}$$

~~of Kan complexes~~

HTPY FIBER PRODUCT

•) (SEGAL) $\forall n \geq 1$, $X_n \xrightarrow[\sim]{w.e.} X_1 \overset{h}{\underset{X_0}{\times}} \cdots \overset{h}{\underset{X_0}{\times}} X_1$

induced from $0<1 \longmapsto i-1<i$, $[1] \to [n]$

eg fiber product if $X_\bullet$ "levelwise fibrant"

•) (COMPLETNESS) X_0 encodes the underlying ∞-groupoid.

RECALL $I[1] = \bullet \overset{\simeq}{\rightleftarrows} \bullet$

Completness can be reformulated as:

$$X_0 = Map^h([0], X_\bullet) \xrightarrow{\simeq} Map^h(N(I[1]), X_\bullet).$$

EXAMPLES: ① If $\mathcal{C}$ is an ordinary category, then $N\mathcal{C}$ satisfies the Segal condition, and if in $\mathcal{C}$ every isomorphism is an identity, $N\mathcal{C}$ is complete.

② Modify to $N(\mathcal{C}, Iso\,\mathcal{C})$ always a complete Segal space [REZK]

③ Given "relative category" $\mathcal{C} \supset W$ (weak equiv)

$\rightsquigarrow N(\mathcal{C}, W)$ also complete Segal space [REZK, BARWICK-KAN, MAZEL-GEE]

4. $\mathrm{Bord}_n^{(\infty,1)}$ [LURIE] IS NOT COMPLETE.

(SEE TOMORROW)

THM (UNICITY, TOËN $(\infty,1)$, BARWICK-SCHOMMER-PRIES (∞,n))

UP TO AN ACTION OF $(\mathbb{Z}/2\mathbb{Z})^n$,

ALL MODELS OF (∞,n)-CATEGORIES ARE EQUIVALENT ~~QUILLEN EQUIV~~ OR $\simeq$ OF $(\infty,1)$-CATEGORIES

USUALLY HAVE A MODEL CATEGORY WHOSE FIBRANT OBJECTS ARE THE GIVEN "$(\infty,1)$-CAT", OR AN $(\infty,1)$-CAT OF SUCH...

HAVE AXIOMS

NOTE: DIRECT EQUIVALENCES IN PRACTICE ARE NOT EASY / SIMPLE.

[RIEHL-VERITY] "MODEL FREE" THEORY OF $(\infty,1)$-CATEGORIES

(∞,n)?