CLAUDIA SCHEIMBAUER, LECTURE 2

HIGHER CATEGORIES

$$\begin{tikzcd} & \text{HIGHER CATEGORIES} \arrow[dl] \arrow[dr] & \\ (Ch, \simeq) & & \left\{\begin{array}{l} Cat \\ Fun \\ \text{nat transf} \end{array}\right. \end{tikzcd}$$

IDEA: USE ENRICHED CATEGORIES

$(\mathcal{S}, \otimes)$ MONOIDAL CAT

DEF: AN $\mathcal{S}$-ENRICHED CATEGORY $\mathcal{C}$ CONSISTS OF

(o) A (SET OF) OBJECTS $Ob\,\mathcal{C}$

(M) $\forall\, X, Y \in Ob\,\mathcal{C}$, AN OBJECT $Hom_{\mathcal{C}}(X,Y) \in \mathcal{S}$

COMPOSITION: $\forall\, X, Y, Z \in Ob\,\mathcal{C}$

$$Hom_{\mathcal{C}}(X,Y) \otimes Hom_{\mathcal{C}}(Y,Z) \longrightarrow Hom_{\mathcal{C}}(X,Z)$$

MAP IN $\mathcal{S}$

IDENTITIES: $\forall\, X \in Obj\,\mathcal{C}$, A MORPH $\mathbb{1} \xrightarrow{id_X} Hom_{\mathcal{C}}(X,X)$ IN $\mathcal{S}$.

SATISFYING ASSOCIATIVITY AND UNITALITY CONDITIONS. (HAVE DIAGRAMS THAT MUST COMMUTE.)*

EX: $(\mathcal{S}, \otimes) = (Set, \times)$ GET BACK SMALL CATEGORIES.

$(\mathcal{S}, \otimes) = (Vect, \otimes)$ GET LINEAR CATEGORIES

$(\mathcal{S}, \otimes) = (Ch, \otimes)$ GET dg-CATEGORIES

① $(\mathrm{Cat}, \times) \rightsquigarrow$ "2-CATEGORY"

② $(\mathrm{Top}, \times) \rightsquigarrow$ TOPOLOGICALLY ENRICHED CATEGORIES

$(\mathrm{sSet}, \times) \rightsquigarrow$ SIMPLICIALLY ENRICHED CAT.

EX of ①: (SYM MON Cat, SYM MON FUNCTORS, SYM. MON NAT. TRANSF.)

$\mathcal{B}, \mathcal{C}$ $\qquad$ $\mathrm{Fun}^{\otimes}(\mathcal{B}, \mathcal{C})$ IS THE CATEGORY OF FUNCTORS AND NAT. TRANSF.

PROBLEM: ★ THE DIAGRAMS IN THE DEF OF ENRICHED CAT. DO NOT COMMUTE ON THE NOSE, BUT ONLY UP TO AN INVERTIBLE 2-MORPH.

$\rightsquigarrow$ DIFFICULT TO HAVE OTHER EXAMPLES THAN THE ABOVE ONE!

"Def": A BICATEGORY IS THE SAME DATA AS A 2-CATEGORY BUT WITH ASSOCIATIVITY AND UNITALITY REPLACED BY 2-ISOMORPHISMS, ~~[illegible]~~

= INVERTIBLE ~~[illegible]~~ ~~[illegible]~~ NAT. TRANSF.

WHICH THEMSELVES SATISFY A COMM. DIAGRAM.

WE DRAW OBJECTS IN A BICATEGORY AS $\bullet$

1-MORPH = OBJ. IN A Hom AS $\bullet \to \bullet$

2-MORPH = MORPH IN A Hom AS
$$\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & Y \end{tikzcd}$$

EXAMPLE OF BICATEGORY

① Alg^{bi} OBJECTS : ALGEBRAS $\ni A, B$

~~MORPHISMS~~

$$\mathrm{Hom}_{Alg^{bi}}(A,B) = \begin{cases} \text{OBJ} : (A,B)\text{-BIMODULES} \\ \text{MORPH} = \text{MORPHISMS OF BIMODULES.} \end{cases}$$

COMPOSITION: ${}_B N_C \circ {}_A M_B = {}_A (M \otimes_B N)_C$

QUESTION: WHY IS THIS NOT A 2-CAT ?

② $Span^{bi}$ = SPANS OF SETS

Obj = Sets

1-MORPH: FROM X TO Y IS PAIRS OF MAPS

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

2-MORPH: ISO CLASSES OF

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & \Sigma \arrow[u] \arrow[d] & Y \\ & S' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

WHERE "ISO CLASSES" MEAN

$$\begin{tikzcd} \Sigma \arrow[r, "\cong"] \arrow[d] \arrow[dr] & \Sigma' \arrow[dl] \arrow[d] \\ S \arrow[d] \arrow[dr] & S' \arrow[dl] \arrow[d] \\ X & Y \end{tikzcd}$$

COMPOSITION IS GIVEN BY TAKING PULLBACKS (INVOLVES CHOICE OF PULLBACKS)

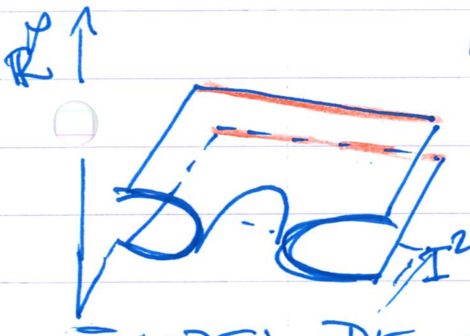
$$\begin{tikzcd} & & S \times_Y T \arrow[dl] \arrow[dr] & & \\ & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

3 BORDISMS [SCHOMMER-PRIES' THESIS]

$2Cob^{ext}$: OBJ = FINITE SETS OF POINTS

(INFORMALLY!) 1-MORPH = 1-dim MFD WITH BDRY, TOGETHER WITH A DIFFEO $\partial M \cong \underbrace{\partial_{in} M}_{\text{SOURCE}} \sqcup \underbrace{\partial_{out} M}_{\text{TARGET}}$

2-MORPH = ISO CLASSES OF 2dim'l "BORDISMS BETWEEN BORDISMS" = 2dim MFDS WITH CORNERS, WHICH CAN BE EMBEDDED IN $[0,1]^2 \times \mathbb{R}^\infty$



TARGET = BACK

SOURCE = FRONT

NOTE: THE OTHER TWO SIDES ARE CYLINDERS FROM

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & B \end{tikzcd} \rightsquigarrow \text{"} \begin{tikzcd} A \arrow[r] \arrow[dr, phantom, "\Uparrow"] & B \\ A \arrow[u, "id"] \arrow[r] & B \arrow[u, "id"'] \end{tikzcd} \text{"}$$

THIS IS A SYMM. MONOIDAL BICATEGORY.

HOW CAN WE INCLUDE k-MORPHISMS FOR ALL $k \geq 1$?

START WITH THE SITUATION WHERE ALL k-MORPHISMS ARE INVERTIBLE: "∞-GROUPOID"

IDEA (GROTHENDIECK)

SPACE X $\rightsquigarrow$ FUNDAMENTAL GROUPOID $\Pi_{\leq 1}(X)$

OBJ = POINTS IN X

MORPH = HTPY CLASSES OF PATHS IN X

INSTEAD: $\rightsquigarrow$ BICATEGORY FUNDAMENTAL 2-GROUPOID $\pi_{\leq 2}(X)$

OBJ: POINTS
1-MORPH: PATHS
2-MORPH: HTPY CLASSES OF HOMOTOPIES

⋮

FUNDAMENTAL ∞-GROUPOID $\pi_{\infty}(X)$

HOMOTOPY HYPOTHESIS: AN ∞-GROUPOID IS A SPACE
OR RATHER HOMOTOPY TYPE ...
i.e. A SPACE UP TO WEAK EQUIV.

IF WE HAVE A DEFINITION OF THE LHS, WE WANT THIS TO BE A THEOREM. OTHERWISE IT IS A DEFINITION...
OR IN OTHER WORDS: A POTENTIAL DEF. OF ∞-GROUPOIDS IS ONLY GOOD IF IT SATISFIES THE HOMOTOPY HYPOTHESIS.
[CAN ALSO DO A FINITE STAGE VERSION:
n-GROUPOIDS $\leftrightarrow$ n-TYPES = SPACES WITH ONLY $\pi_0, \dots, \pi_n \neq 0$]

SO NOW WE CAN TRY TO DEFINE $(\infty,1)$-CATEGORY $\mathcal{C}$

k-MORPH $\forall k$

FOR $k \geq 1$, k-MORPH ARE INVERTIBLE.

FOR X, Y OBJ. IN $\mathcal{C}$,
WANT $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS AN ∞-GROUPOID,
i.e. A SPACE.
(HOMOTOPY HYP)

Def (Attempt 1) An **$(\infty,1)$-cat.** is a category enriched in Spaces.

Problem: Too strict.

Ex: **$I[1]$**: Obj, Morph

$$\bullet \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \bullet \xrightarrow{\simeq} \bullet$$

Groupoid — Equiv. as categories