Copenhagen Master Class, August 2021 Claudia Scheimbauer

Dualizabitility, higher categories, and topological field theories

This is a long selection of exercises of very different levels and with motivations coming from different areas. I am aware that this list is too long for the problem session(s). Pick the one(s) you find interesting and look up or ask for the precise definitions if needed.

(1) Find the dualizable objects in the following monoidal categories:

  (a) vector spaces and direct sum,

  (b) vector spaces and tensor product,

  (c) pointed vector spaces (a vector space together with a chosen vector in it), point-preserving linear maps, and tensor product,

  (d) sets and cartesian product,

  (e) Span, where objects are sets, a morphism from X to Y is an isomorphism class of spans $X \leftarrow S \rightarrow Y$, composition is pullback, and the monoidal product is the cartesian product,

  (f) Alg, where objects are $\mathbb{C}$-algebras, a morphism from an algebra A to an algebra B is an isomorphism class of bimodules, composition is relative tensor product,

$$ {}_BN_C \circ_A M_B =_A M_B \otimes_B {}_BN_C, $$

  and tensor product over $\mathbb{C}$ as the monoidal structure,

  (g) nCob and disjoint union.

(2) Show that if Z is an n-dimensional topological field theory, then for any closed $(n-1)$-dimensional manifold, $Z(M)$ is finite dimensional.

(3) Let $\mathcal{B}, \mathcal{C}$ be symmetric monoidal categories. Assume that every object in $\mathcal{B}$ has a dual. Show that $\mathrm{Fun}^{\otimes}(\mathcal{B}, \mathcal{C})$ is a groupoid. What can you conclude for TFTs?

(4) Work through the details showing that $\mathrm{Fun}^{\otimes}(1\mathrm{Cob}^{or}, \mathcal{C}) \simeq \mathcal{C}^{\mathrm{dualizable}}$.

(5) Can you modify the definition of the 1-dimensional cobordism category so that if $\mathcal{C}$ is (braided) monoidal, then we get the characterization of (braided) monoidal functors into $\mathcal{C}$ via objects which have a left and right dual in $\mathcal{C}$?

(6) Can you find a target symmetric monoidal category so that singular homology gives an (oriented/unoriented/...) nTFT into it? (Hint: You may want to try modifying (**??**).)

(7) Which "different" framings can you find on $* \in 1\mathrm{Cob}^{fr}$ and on $S^1 \in 2\mathrm{Cob}^{fr}$?

(8) Look up the details of the definition of a quasi-category. Show the following properties:

  (a) Translate the horn-filling conditions for Kan complexes and quasi-categories in dimensions 1, 2, and 3 into categorical content.

  (b) Prove that $N\mathcal{C}$ is a quasicategory.

  (c) Let $\tau_1 \colon sSet \rightarrow Cat$ be the left adjoint to the nerve functor, called *homotopy category*. Work out/look up an explicit description of τ_1.

(9) Show that $N\mathcal{C}$ is always a Segal space. Show that it is complete iff every isomorphism is an identity.

(10) Show that the 2-fold simplicial space $\mathrm{Bord}_2^{(\infty,2)}$ as defined in class is essentially constant by unravelling the submersivity condition. How could you define an (∞, k)-category of n-dimensional bordisms, for any $k \geq 1$?

(11) (a) Convince yourself that an E_1-algebra in $\mathcal{S} = (\mathrm{Vect}, \otimes)$ is the same data as an associative algebra. Use that Vect is an ordinary category.

  (b) Compute/guess $\int_{S^1} A$. You may want to use the explicit formula for computing a left Kan extension as a colimit.

(12) (a) Let A be an associative algebra viewed as an object in $(\mathrm{Vect}, \otimes)$. Compute/guess $\int_{S^1} A$. Here you may want to use that a symmetric monoidal functor $\mathcal{D}\mathrm{isk}_1^{fr} \to (\mathrm{Vect}, \otimes)$ is the same as a symmetric monoidal functor $\mathrm{Disk}_1^{or} \to (\mathrm{Vect}, \otimes)$, because $(\mathrm{Vect}, \otimes)$ is an ordinary category.

(b) Now view A as an associative algebra viewed as an object in $(\mathrm{Ch}, \otimes)$. What changes?

(13) Show that a $\mathcal{D}\mathrm{isk}_1^{or}$-algebra in $(\mathrm{Cat}, \times)$ is equivalent to a (unital) monoidal category.

(14) Show that a constructible factorization algebra on $(0, 1) \supset \{s\}$ is a bimodule with a distinguished point. What consequences does this have for the morphisms?

(15) Show that the following is a complete Segal space: $\mathrm{Int}_\bullet \colon \Delta^{op} \to \mathrm{Top}$,

$$\mathrm{Int}_k = \{I_0 \leq \cdots \leq I_k\} \subset \mathbb{R}^{k+1},$$

where $I_0 \leq \cdots \leq I_k$ as in class, i.e. $a_0 \leq \cdots \leq a_k$ and $b_0 \leq \cdots \leq b_k$ and for every i we have that $a_i < b_i$. Note that we have a forgetful map $\mathrm{Bord}_n^{(\infty,1)} \to \mathrm{Int}$.

(16) Show that in $\mathcal{C} = \mathrm{Alg}_1(\mathcal{S})$ as defined in class every object has a dual. For this you need to know that the monoidal structure on $\mathcal{C}$ is given (informally) by taking the tensor product of factorization algebras, where $\mathcal{F} \otimes \mathcal{G}(U) = \mathcal{F}(U) \otimes \mathcal{G}(U)$; possibly this involves first moving the point in the stratification by a linear rescaling.

(17) Work out what the objects, 1-morphisms, and 2-morphisms in $\mathrm{Alg}_2(\mathcal{S})$ should be, using constructible factorization algebras on suitable stratifications on $(0, 1)^2$. Unravel (informally) what this amounts to for $(\mathrm{Vect}, \otimes)$. Using Adrien's explanation for E_2-algebras in $(\mathrm{Cat}, \times)$, can you unravel what 1-morphisms are?

(18) In the 2-dimensional setting, compute factorization homology of the torus using either excision or the fully extended 2-dimensional TFT by decomposing the torus in the bordism category.

(19) (a) Let Ch be the $(\infty, 1)$-category of cochain complexes, but with direct sum as the symmetric monoidal structure. Show that any cochain complex V admits an E_n-algebra structure given by defining the coproduct to be the addition map. Show further that this structure is unique.

(b) For $(\mathrm{Ch}, \oplus)$, fix A any abelian group (considered as a cochain complex in degree 0). By the previous exercise, addition endows A with an E_n- structure for any n. Show that for any framed manifold M, there is a quasi-isomorphism of cochain complexes

$$\int_M A \simeq C^*(M, A)$$

is the cochain complex of singular chains. (Hint: Show that excision for $\oplus$ gives rise to the Mayer-Vietoris sequence; this is a consequence of the usual excision theorem for singular homology. To be precise, for this you should show that $\oplus$ preserves sifted colimits in each variable.

(20) Show that every object in $\mathrm{Bord}_2^{(fr)}$ is 2-dualizable. That is, show that every object has a dual and the evaluation and coevaluation maps have left and right adjoints.

(21) Let $\mathcal{C}$ be monoidal and let $X \in \mathcal{C}$ be a dualizable object. Show that if (Y, c, u) and (Y', c', u') are triples of dualizability data (i.e. a dual, an evaluation map, and a coevaluation map), then there is a unique map $(Y, c, u) \to (Y', c', u')$.

(22) (Open problem.) Can you classify the E_n-algebras in $(Mfld_n^{fr}, \amalg)$?

(23) Find your own question to ask about factorization homology!