Copenhagen Master Class, August 2021 Claudia Scheimbauer

Dualizabitility, higher categories, and topological field theories

This is a long selection of exercises of very different levels and with motivations coming from different areas. I am aware that this list is too long for the problem session(s). Pick the one(s) you find interesting and look up or ask for the precise definitions if needed.

(1) Find the dualizable objects in the following monoidal categories:

  (a) vector spaces and direct sum,

  (b) vector spaces and tensor product,

  (c) pointed vector spaces (a vector space together with a chosen vector in it), point-preserving linear maps, and tensor product,

  (d) sets and cartesian product,

  (e) Span, where objects are sets, a morphism from X to Y is an isomorphism class of spans $X \leftarrow S \to Y$, composition is pullback, and the monoidal product is the cartesian product,

  (f) Alg, where objects are $\mathbb{C}$-algebras, a morphism from an algebra A to an algebra B is an isomorphism class of bimodules, composition is relative tensor product,

$$ {}_BN_C \circ_A M_B = {}_A M_B \otimes_B {}_BN_C, $$

  and tensor product over $\mathbb{C}$ as the monoidal structure,

  (g) nCob and disjoint union.

(2) Show that if Z is an n-dimensional topological field theory, then for any closed $(n-1)$-dimensional manifold, $Z(M)$ is finite dimensional.

(3) Let $\mathcal{B}, \mathcal{C}$ be symmetric monoidal categories. Assume that every object in $\mathcal{B}$ has a dual. Show that $\mathrm{Fun}^{\otimes}(\mathcal{B}, \mathcal{C})$ is a groupoid. What can you conclude for TFTs?

(4) Work through the details showing that $\mathrm{Fun}^{\otimes}(1\mathrm{Cob}^{or}, \mathcal{C}) \simeq \mathcal{C}^{\mathrm{dualizable}}$.

(5) Can you modify the definition of the 1-dimensional cobordism category so that if $\mathcal{C}$ is (braided) monoidal, then we get the characterization of (braided) monoidal functors into $\mathcal{C}$ via objects which have a left and right dual in $\mathcal{C}$?

(6) Can you find a target symmetric monoidal category so that singular homology gives an (oriented/unoriented/...) nTFT into it? (Hint: You may want to try modifying (1f).)

(7) Which "different" framings can you find on $* \in 1\mathrm{Cob}^{fr}$ and on $S^1 \in 2\mathrm{Cob}^{fr}$?

(8) Look up the details of the definition of a quasi-category. Show the following properties:

  (a) Translate the horn-filling conditions for Kan complexes and quasi-categories in dimensions 1, 2, and 3 into categorical content.

  (b) Prove that $N\mathcal{C}$ is a quasicategory.

  (c) Let $\tau_1 \colon sSet \to Cat$ be the left adjoint to the nerve functor, called *homotopy category.* Work out/look up an explicit description of τ_1.

(9) Show that $N\mathcal{C}$ is always a Segal space. Show that it is complete iff every isomorphism is an identity.

(10) Show that the 2-fold simplicial space $\mathrm{Bord}_2^{(\infty,2)}$ as defined in class is essentially constant by unravelling the submersivity condition. How could you define an (∞, k)-category of n-dimensional bordisms, for any $k \geq 1$?

(11) (a) Convince yourself that an E_1-algebra in $\mathcal{S} = (\mathrm{Vect}, \otimes)$ is the same data as an associative algebra. Use that Vect is an ordinary category.

  (b) Compute/guess $\int_{S^1} A$. You may want to use the explicit formula for computing a left Kan extension as a colimit.