BROCHIER 7

GOAL FOR TODAY: WE CAN COMPUTE $\int_S A$ WHEN A IS BALANCED, BRAIDED MONOIDAL.

WHY? (IS IT INTERESTING) 1) THIS IS ONE OF THE FIRST NON-TRIVIAL EXAMPLES OF FACTORIZATION HOM.
$\rightsquigarrow$ BRAID GROUP REPS, TQFT, ...

2) ONE CAN GET SURFACE BRAID GROUPS REPS, INVARIANTS OF LINKS IN $S \times I$, AND MAPPING CLASS GROUPS.

3) $A = \mathrm{Rep}_q G$ (MAIN EXAMPLE THAT WE KNOW EXPLICITLY)

$\mathrm{Rep}\, G$ SYMM. MONOIDAL $\rightsquigarrow$ $\mathrm{Rep}_q G$ BRAIDED
DEFORMATION

LOOPS ON S (HTPY INV) $\rightsquigarrow$ LINKS IN $S \times I$ (ISOTOPY INV.)

$S \in \mathrm{Mnfd}_2$

$$\emptyset = \text{UNIT} \xhookrightarrow[\text{UNIQUE}]{} S$$

$\downarrow \int$

$$\begin{cases} \mathrm{Vect} \longrightarrow \int_S A & \text{FUNCTOR IN LFP} \\ \mathbb{C} \longmapsto \mathcal{O}_S & \rightsquigarrow \int_S A \text{ is POINTED.} \end{cases}$$

IF $A = \mathrm{Rep}\, G$, $x \in S$, $\mathcal{O}_S \cong \mathcal{O}(R(S,x))$ AS AN OBJECT IN $\mathcal{O}(R(S,x))\text{-mod}_G$

$\mathcal{O}_S$ is A STRUCTURE SHEAF (CAN BE IDENTIFIED AS SUCH)

$a \in A$ IS AN ALGEBRA

$\iff \exists\, m : a \otimes a \longrightarrow a$ ASSOCIATIVE

$m \circ (m \otimes id) = m \circ (id \otimes m)$

$a\text{-mod}_A$ = CATEGORY OF a-MODULES IN A

$v \in A$ s.t. $\exists\, a \otimes v \to v$

...

<u>THM</u> (BZBJ 1 AND 2) FOR A RIGID (EVERY COMPACT OBJ HAS A LEFT AND RIGHT DUALS)

1) ASSUME $\partial S \neq \emptyset$, $I \subset \partial S$. THERE IS A CANONICAL ALGEBRA STRUCTURE ON $\mathcal{O}_S$ AND A CANONICAL EQUIVALENCE

$$\int_S A \cong \mathcal{O}_S\text{-mod}_A \quad (*)$$

CORRESPOND TO PICK $x \in I$ $\rightsquigarrow$ $\mathcal{O}C(Ch(S)) \cong \underset{\mathcal{O}_S \text{ IN THAT CASE}}{\mathcal{O}(R(S,x))}\text{-mod}_G$

FOR $A = \mathrm{Rep}\, G$

WAS A DEF FOR US BUT CAN BE MADE INTO A STATMENT...

2) IF S is CLOSED, $\mathring{S} = S \setminus D$,

$\exists\, \mu_S : \mathcal{O}_{Ann} \longrightarrow \mathcal{O}_{\mathring{S}}$ AND

$Ann = S^1 \times I$

$\int_S A \cong$ CERTAIN CATEGORY OF $\mathcal{O}_{\mathring{S}}\text{-mod}_A$ ON WHICH $\mathcal{O}_{Ann}$ ACTS TRIVIALLY (THROUGH μ).

[Corresponds to

$\mu_{\mathring{S}}: R(\mathring{S}, x) \longrightarrow R(\mathrm{Ann})$ moment map

$Ch(S) = \mu^{-1}(\{id\}) / G$]

3) Let $\partial S \neq \emptyset$, $I \subset \partial S$

Every choice of a skeleton with one vertex (on I) gives a "presentation by generators and relations" of $\mathcal{O}_S$.

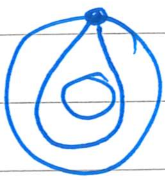
[Analogue of $R(S, x) \cong G^{2g+n-1}$]

$\rightsquigarrow$ "explicit" description of $\int_S A$, at least as explicit as we know A...

Example: $S = \mathrm{Ann}$

[$S = D \rightsquigarrow \mathcal{O}_D = 1 \in A = \int_D A$]

Skeleton



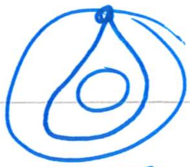
(*) Note: $\emptyset \hookrightarrow D \hookrightarrow S$

$$\begin{tikzcd} \int_\emptyset A = \mathrm{Vect} \arrow[r] & A = \int_D A \arrow[r] & \int_S A \arrow[l, bend right] \\ \mathbb{C} \arrow[r, mapsto] & 1 \arrow[r, mapsto] & \mathcal{O}_S \end{tikzcd}$$

$\exists$ right adjoint R_S that is cocontinuous (i.e. preserves colimits) and faithful.

$\mathcal{O}_S \equiv R_S(\mathcal{O}_S) \in A$

↑ we identify $\mathcal{O}_S$ with its image in A.

BACK TO  : AS AN OBJECT IN $\mathcal{A}$,

$$\underset{\substack{\| \\ \mathrm{Ann}}}{\mathcal{O}_{S^1\times I}} := \mathcal{O}_{\mathcal{A}} = \underbrace{\int}_{\text{COEND}}^{x \in \text{COMPACT OBJ.}} x^* \otimes x$$

FOR $\mathcal{A}$ SEMI-SIMPLE $\mathcal{O}_{\mathcal{A}} = \bigoplus_{\substack{\text{SIMPLE} \\ \text{COMPACT} \\ x}} x^* \otimes x$

($\leftrightarrow$ PETER-WEYL DECOMPOSITION $\mathcal{O}(G) = \bigoplus V^* \otimes V$)

MULTIPLICATION:

$$\begin{array}{ccc} (x^*\otimes x)\otimes(y^*\otimes y) & & \mathcal{O}_{\mathcal{A}}\otimes\mathcal{O}_{\mathcal{A}} \\ \downarrow & & \downarrow \\ y^*\otimes x^*\otimes x\otimes y & & \\ \| & & \\ (x\otimes y)^*\otimes(x\otimes y) & & \mathcal{O}_{\mathcal{A}} \,. \end{array}$$

NOTE: THIS ALGEBRA IS WELL-KNOWN, AT LEAST WHEN $\mathcal{A} = \mathrm{Rep}\, G$ OR $\mathrm{Rep}_q G$. [DONIN-KULISH-MUDOV] =[DKM]

EX: FOR $\mathrm{Rep}_q G$, THEN BY DONIN-MUDOV, AND ALEKSEEV-GROSSE-SCHOMERUS, THIS ALGEBRA QUANTISES $G_{STS} \simeq R(\mathrm{Ann})$.

EVERY COMPACT OBJECT IN $\mathcal{A}$ IS A COMODULE OVER $\mathcal{O}_{\mathcal{A}}$:

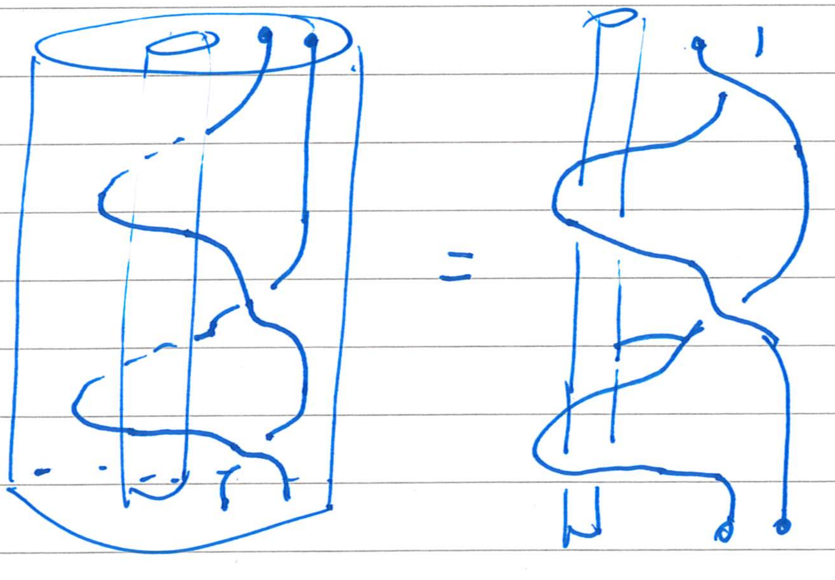
$$\begin{tikzcd} & x \arrow[d] & \\ x \otimes & x^* \otimes & x \arrow[d] \\ x \arrow[d] & & \mathcal{O}_A \\ x \otimes & \mathcal{O}_A & \end{tikzcd}$$

$\rightsquigarrow$ CAN DEFINE A MAP

$$X_x \quad \begin{tikzcd} x \otimes \mathcal{O}_A \arrow[d] \\ x \otimes \mathcal{O}_A \otimes \mathcal{O}_A \arrow[d, "\otimes\, id \otimes m"] \\ x \otimes \mathcal{O}_A \end{tikzcd}$$

<u>PROP</u> [DKM, MAB) X_x SATISFIES THE REFLECTION EQUATION

$$\beta_{y,x} X_y \beta_{x,y} X_x = X_y \beta_{y,x} X_x \beta_{x,y}$$



DEFINING RELATION FOR $B_n(A_{nn})$
''
BRAID GROUP OF ANNULUS.

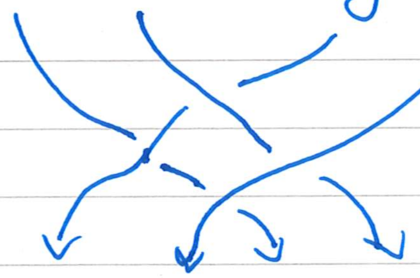
[FOR $x=y=V$, $A=Rep_q GL_2$, THIS IS ALSO DEFINING RELATION FOR $\mathcal{O}_{Rep_q GL_2}$]

EX: $T^2 \setminus D$, $\mathcal{O}_{T^2\setminus D} =$ "D_A" $= \mathcal{O}_A \otimes \mathcal{O}_A$

AND EACH COPY OF $\mathcal{O}_A$ IS A SUBALG.

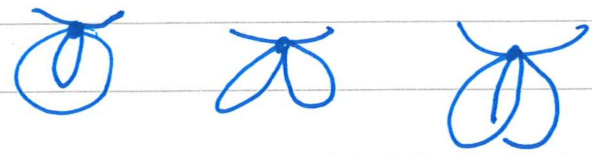
CROSS RELATION: $\mathcal{O}_A \otimes \mathcal{O}_A$

$x^* \otimes x \otimes y^* \otimes y$

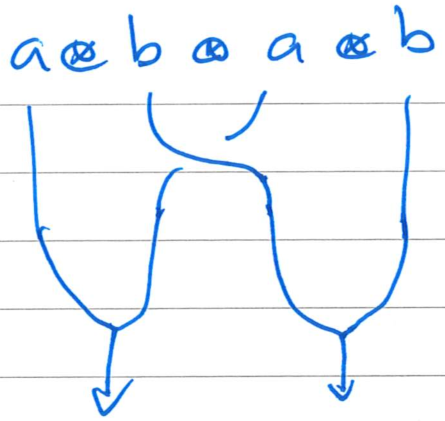


MORE GENERALLY, IF A SKELETON OF S (WITH 1 VERTEX) HAS k EDGES, THEN $\mathcal{O}_S \cong \mathcal{O}_A^{\otimes k}$ WITH A BRAID FOR "SWITCHING COPIES" = CROSS RELATION DEPENDING OF HOW THE EDGES ARE ORDERED AT THE VERTEX. THAT ALLOWS TO DEFINE THE MULTIPLICATION.

FOR EVERY PAIR OF EDGES THERE ARE THREE POSSIBILITIES:



LET a, b BE ~~AN~~ ALGEBRAS IN A. $a \tilde{\otimes} b$ IS AN ALGEBRA VIA



"BRAIDED TENSOR PRODUCT"

<u>Claim</u> $\displaystyle\int_{\Sigma_{g,n}} A \cong \mathcal{D}_A^{\tilde{\otimes} g} \,\tilde{\otimes}\, \mathcal{O}_A^{\tilde{\otimes} n-1}\text{-mod}_A$, $n>0$

IN PARTICULAR, THE BRAIDED TENSOR PRODUCT QUANTISES FUSION OF G-POISSON SPACES.