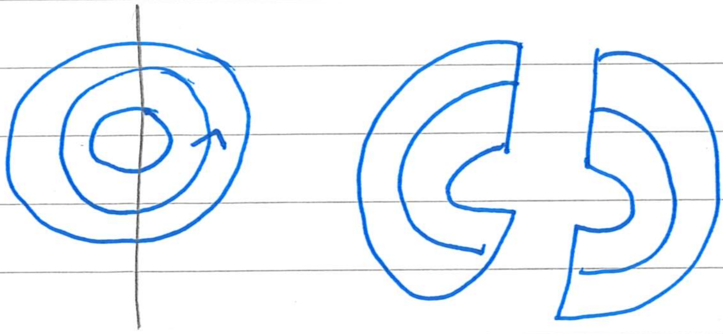
ADRIEN BROCHIER, LECTURE 5



NEED SUCH OBJECTS IF WE WANT TO BE ABLE TO CUT

SLOGAN: FACTORIZATION HOMOLOGY IS A FANCIER VERSION OF THE SKEIN CATEGORY THAT GIVES ~~THE~~ MORE CORRECT ANSWER/MODEL...

REPLACE $\mathrm{Rep}\,G \rightsquigarrow \mathrm{Rep}_q\,G$ | AND WORKS MORE GENERALLY.

3) CATEGORIFYING LINEAR ALGEBRA

EVERY CATEGORY HERE ~~IS~~ $\mathbb{C}$-LINEAR.

LET $\mathcal{C}$ BE A ($\mathbb{C}$-LINEAR) CATEGORY.

<u>DEF</u>: $x \in \mathcal{C}$ IS CALLED <u>COMPACT</u> IF $\mathrm{Hom}(x,-)$ PRESERVES FILTERED COLIMITS. (eg. INFINITE $\oplus$) AND <u>PROJECTIVE</u> IF $\mathrm{Hom}(x,-)$ PRESERVES FINITE COLIMITS, IS <u>TINY</u> IF $\mathrm{Hom}(x,-)$ COMMUTES WITH ALL SMALL COLIMITS.

<u>EX</u>: A ALGEBRA. $\mathcal{C} = A\text{-mod}$.

COMPACT $\iff$ FINITELY GENERATED.

TINY $\iff$ FIN. GEN. PROJECTIVE.

DEF: $\mathcal{C}$ is called $\underset{\text{LFP}}{\underline{\text{locally finitely presentable}}}$ if it has all small colimits and every object is a filtered colimit of compacts. ("basis for the category")

$\mathcal{C}$ is locally tiny if it has all small colimits and ___ ... ___ colimit of tiny objects.

eg: • A-Mod is locally tiny. Generating set $S = \{A\}$

• C a coalgebra. C-Comod is locally finitely presentable with $S = \{\text{f.d. comodules}\}$

• $\mathcal{O}(G)$-comod = Rep(G) is locally tiny.

Let LFP be the (2,1)-category whose objects are LFP categories and morphisms are linear, colimit preserving, functors, 2-morph are natural isomorphisms.

LFP_c = full subcategory functors also preserve compact objects.

"Adjoint functor theorem" 1) A functor between two LFP-categories is colimit preserving iff it has a right adjoint.

2) IT PRESERVES COMPACT OBJECTS IFF THE RIGHT ADJOINT PRESERVES FILTERED COLIMITS.

Def: (THE DELIGNE-KELLY TENSOR PRODUCT)
LET $\mathcal{C}, \mathcal{D} \in LFP_{(c)}$. THE TENSOR PRODUCT $\mathcal{C} \boxtimes \mathcal{D}$ IS AN $LFP_{(c)}$ CATEGORY, UNIVERSAL FOR THE FOLLOWING:

$$\left\{ \begin{array}{l} \text{BILINEAR FUNCTORS,} \\ \text{COLIMIT PRESERVING} \\ \text{IN EACH VARIABLE} \end{array} \;\; \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E} \right\}$$

$$\text{IS}$$

$$\left\{ \mathrm{Hom}(\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E}) \right\}$$

PROPOSITION: THIS EXISTS.

$(LFP_{(c)}, \boxtimes)$ IS A (CLOSED) SYMM. MONOIDAL (2,1)-CATEGORY.

EX: $A\text{-Mod} \boxtimes B\text{-Mod} \simeq (A \otimes B)\text{-Mod}$.

Def: A MONOIDAL CATEGORY (FOR THE PURPOSE OF THIS LECTURE!) IS AN E_1-ALGEBRA IN LFP.

$\Leftrightarrow$ IT'S A MONOIDAL CATEGORY $\mathcal{A}$ IN THE USUAL SENSE S.t. $T : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathcal{A}$ IS A MORPHISM IN LFP.

Def: A BRAIDED MONOIDAL CATEGORY (FOR THE PURPOSE OF THIS TALK) IS AN E_2-ALGEBRA IN LFP.

$\Leftrightarrow$ BRAIDED MONOIDAL CATEGORY IN THE USUAL SENSE SUCH THAT THE MONOIDAL CAT IS AS ABOVE.

REFS: [Fr, BCJF].

3. (n+1) FACTORIZATION HOMOLOGY

LET Mfd_2^{or} BE THE (2,1)-CATEGORY WITH OBJECTS ORIENTED SURFACES, AND

$$Hom(M,N) = \Pi_1(Emb^{or}(M,N))$$

$\Uparrow$
FUNDAMENTAL GROUPOID
= EMBEDDINGS + ISOTOPIES
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ||
$\quad\quad\quad\quad\quad\quad\quad$ PATHS OF EMBEDDINGS

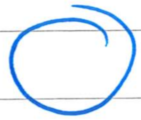
$Disk_2^{or} \subset Mfd_2^{or}$
||
FULL SUBCATEGORY WHOSE OBJECTS ARE FINITE DISJOINT UNIONS OF DISCS.

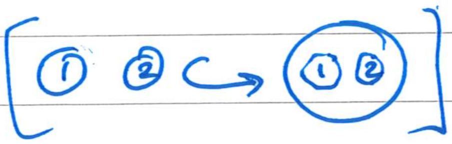
BOTH CATEGORIES ARE SYM. MONOIDAL wrt $\amalg$.

Claim: A SYM. MONOIDAL FUNCTOR $Disks_2^{or} \to LFP$ IS "THE SAME AS" A (BALANCED) BRAIDED

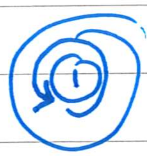
MONOIDAL CATEGORY.

(IN PARTICULAR, RIBBON CATEGORIES ARE BALANCED.)

IDEA:  $\longmapsto A$

 $\longmapsto \left[A \boxtimes A \xrightarrow{T} A \right]$

 $\longmapsto \left[\beta : X \otimes Y \cong Y \otimes X \right]$
BRAIDING

 $\longmapsto \left[\theta : X \longrightarrow X \right]$
BALAN

NOTE: THIS REALLY COMES FROM A FUNCTOR FROM A SMALLER EQUIVALENT SUBCATEGORY OF Disk_2^{or} WHERE WE CHOOSE A FEW BASEPOINTS IN $\pi_1(\mathrm{Emb}(-)) \leftrightsquigarrow$ BRAID/RIBBON OPERAD

THM: ([AF], [LU]) + LECTURE NOTES BY [TANAKA + Co.]

A BALANCED BRAIDED MONOIDAL.

THERE EXISTS A CANONICAL EXTENSION

$$\begin{tikzcd} \mathrm{Disk}_2^{or} \arrow[r, "A"] \arrow[d, hook] & \mathrm{LFP} \\ \mathrm{Mfd}_2^{or} \arrow[ur, dashed, "\int A"'] & \end{tikzcd}$$

WHICH IS SYMM. MONOIDAL AND SATISFIES, AND IS CHARACTERIZED, BY EXCISION (= COMPATIBILITY WITH GLUING)