ADRIEN BROCHIER, LECTURE 4

2) POISSON STRUCTURE

RECALL THAT A POISSON STRUCTURE ON A COMMUTATIVE ALGEBRA A IS A BILINEAR MAP

$$\{,\} : A \otimes A \longrightarrow A$$

S.T. • $(A, \{,\})$ IS A LIE ALGEBRA

• $\{f, gh\} = \{f, g\}h + \{f, h\}g$

(DERIVATION OF THE PRODUCT = LEIBNIZ RULE)

THM [ATIYAH-BOTT, GOLDMAN]

$Ch(S)$ HAS A CANONICAL POISSON STRUCTURE

REM: REALLY DEPENDS ON S, EG THE PAIR OF PANTS AND PUNCTURED TORUS HAVE SAME π_1, HENCE SAME $Ch(S)$, BUT DISTINCT POISSON STRUCTURES.

(CHARACTER STACK $\underline{Ch(S)} = \mathrm{Map}(S, BG)$

THM ((C) PTVV) HAS A "STACKY POISSON STRUCTURE"

[Sa] SEE ALSO)

2.1) GOLDMAN BRACKET

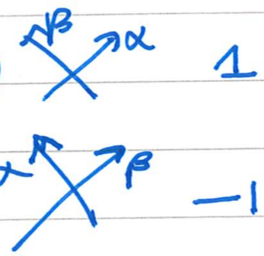
THM (GOLDMAN) THE SPACE OF FORMAL LINEAR COMBINATIONS OF LOOPS ON S HAS A CANONICAL LIE ALGEBRA STRUCTURE

$$\{\alpha,\beta\} = \sum_{p\in\alpha\cap\beta} \varepsilon_p(\alpha,\beta)\,\overline{\alpha \underset{p}{*} \beta}$$

α,β TRANSVERSE

WHERE $\overline{\alpha \underset{p}{*} \beta}$ = CONCATENATION AT p + FORGET THE BASEPOINT p

AND $\varepsilon_p(\alpha,\beta)$ = SIGN



⇝ EXTENDS USING THE LEIBNIZ RULE TO A POISSON BRACKET ON $\mathcal{S}_S$.

PROP: $\mathcal{S}_S \longrightarrow \mathcal{O}(Ch(S))$ IS POISSON

(I.E. THE KERNEL IS A POISSON IDEAL SO THE MAP DEFINES A POISSON STRUCTURE ON THE RHS, THAT CAN ALSO BE DEFINED INDEPENDENTLY).

2.2) COMBINATORIAL (FOCK-ROSLY, [AKS] [AMM], [LBS], ..., [AU])

THERE IS A POISSON STRUCTURE ON G (I.E. ON $\mathcal{O}(G)$) THAT TURNS INTO A POISSON ALGEBRAIC GROUP (DRINFELD):

$m: G \times G \longrightarrow G$ IS A POISSON MAP

i.e. $\mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes \mathcal{O}(G)$ IS A MAP OF POISSON ALGEBRAS.

(THE ORIGINAL DEFINITION OF $[A B]$ DEPENDS ONLY ON THE PAIRING ON $\mathfrak{gl}_n$ $(A, B) \longmapsto \mathrm{tr}(A, B)$

$\rightsquigarrow$ CANONICAL ELEMENT $t = \sum E_{ij} \otimes E_{ji}$

FORMULA FOR THE POISSON STRUCTURE INVOLVES THE CLASSICAL R-MATRIX $r = 2 \sum_{i>j} E_{ij} \otimes E_{ji} + \sum_i E_{ii} \otimes E_{ii}$

$$\frac{1}{2}(r + r^{2,1}) = t$$

$r^{2,1}$ = r WITH TENSOR FACTORS FLIPPED

CAN DEFINE THE BRACKET FROM r AND POISSON RELATION $\iff$ r SATISFIES THE YANG–BAXTER EQUATION

$$[r^{1,2}, r^{1,3}] + [r^{1,2}, r^{1,3}] + [r^{1,3}, r^{2,3}] = 0 \quad \text{in } \mathfrak{gl}_n^{\otimes 3})$$

IF X IS A POISSON VARIETY, AND $G \curvearrowright X$, THE ACTION IS CALLED POISSON IF $X \times G \to X$ IS POISSON ⚠ USING ALSO THE POISSON STRUCTURE OF G.

LET S BE A NOT NECESSARILY CONNECTED SURFACE. LET $x_1, \dots, x_k$ BE IN S WITH AT LEAST ONE POINT IN EACH COMPONENT.

(15)

AN n-di REPR. OF $\Pi = \Pi_1(S, x_1 \ldots x_\ell)$

IS A COPY V_i OF V FOR EVERY x_i AND

FOR EVERY $\underset{\text{PATH}}{\gamma : x_i \longrightarrow x_j} \rightsquigarrow \varphi(\gamma) : V_i \to V_j$

COMPATIBLE WITH COMPOSITION.

$G^\ell \circlearrowright Rep(\Pi)$ i.e. $\left[\begin{array}{l} \gamma : x_i \longrightarrow x_j \\ (g_1, \ldots, g_\ell) \in G^\ell \end{array}\right]$

$$\Big\downarrow$$

$$\varphi(\gamma) \longmapsto g_i^{-1} \varphi(\gamma) g_j$$

A SKELETON ON S IS A ORIENTED GRAPH $(\mathcal{V}, \mathcal{E})$ ON S s.t. $\mathcal{V} \subset \partial S \ (\neq \emptyset)$, EACH COMPONENT HAS AT LEAST ONE VERTEX AND S DEFORMATION RETRACTS ON S.

(THE ORIENTATION OF S INDUCES A CANONICAL ORDERING OF THE HALF-EDGES AT EACH VERTEX.)

A SKELETON GIVES $Ch(S) \cong G^{|\mathcal{E}|} / G^{|\mathcal{V}|}$

<u>THM</u> ([FP], ...) THE CHOICE OF A SKELETON INDUCES A POISSON STRUCTURE ON

$R(\Pi_1(S, \mathcal{V})) \cong G^{|\mathcal{V}|}$ (VIA SOME COMBINATORIAL FORMULA...)

s.t. THE ACTION OF $G^{|\mathcal{V}|} \circlearrowright G^{|\mathcal{E}|}$ IS POISSON

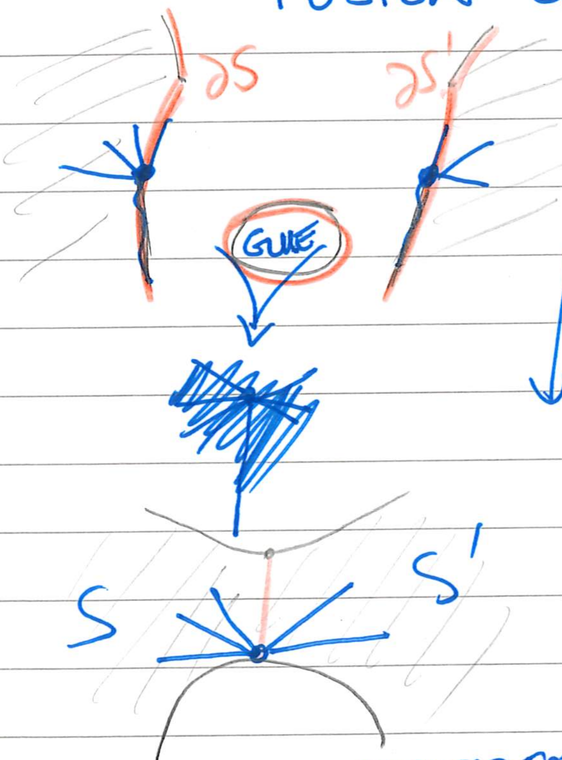
SUCH THAT THE QUOTIENT POISSON STRUCT ON $Ch(S)$ COINCIDES WITH THE AB-G ONE.

IF X IS A $G\times G$-POISSON VARIETY,

$\rightsquigarrow$

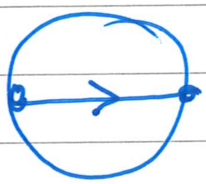
G-POISSON VARIETY VIA DIAGONAL ACTION OF G (MODIFYING APPROPRIATELY THE POISSON STRUCTURE OF X BECAUSE $G \to G\times G$ IS NOT POISSON)

"FUSION OF X" $Fus(X)$



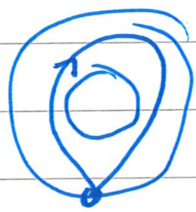
FUSION ↓

GLUING ALONG INTERVALS CORRESPONDS TO THIS FUSION OPERATION.



$\cong G$ POISSON ALGEBRAIC STRUCTURE

GLUE $\rightsquigarrow$ $G\times G$ - POISSON MANIFOLD

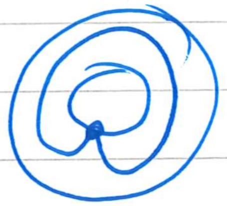


$\cong G = Fus(G)$ G-POISSON MFD

(STS-BRACKET)

CLOSED SURFACES

IF S IS A CLOSED SURFACE, $\mathring{S} = S \setminus D$
WITH BASEPOINT ON $\partial \mathring{S}$

 $\hookrightarrow \mathring{S}$ ~~PRESERVING THE SKELETON~~

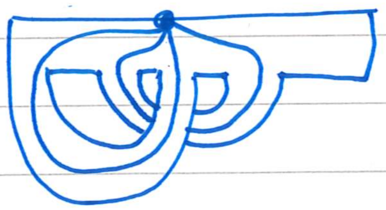
FOR ANY CHOICE OF SKELETON ON $\mathring{S}$,

$R(\mathring{S}) \longrightarrow R($  $)$ IS POISSON

"MOMENT MAP"

$\Downarrow$ $Ch(S) = \mu^{-1}(\{id\})/G$ IS POISSON

$\rightsquigarrow$ ABG.

 $\rightsquigarrow$ POISSON STRUCTURE ON $G \times G$
"HEISENBERG DOUBLE"
"POISSON GROUP VERSION OF THE COTANGENT BUNDLE"