Adrien Brochier, Lecture 3

<u>Recap:</u> Let V be n-dim$/\mathbb{C}$ $\quad G = GL(V)$
S surface

$\mathcal{S}_S$ = polynomial algebra on the set of homotopy classes of (free) loops on S.
= formal $\mathbb{C}$-linear combinations of unions of loops on S.

$$\begin{aligned} \mathcal{S}_S &\longrightarrow \mathcal{O}(Ch(S)) \\ \gamma &\longmapsto [\rho \longmapsto tr(\rho(\tilde{\gamma}))] \end{aligned}$$

surjective algebra map.

<u>Kernel?</u> Remember the analogy

$\rightsquigarrow \varphi \in \mathcal{O}(Ch(S))$

$$\varphi : \rho \longmapsto \begin{array}{ccc} \mathbb{C} & & 1 \\ \downarrow & & \downarrow \\ V \otimes V^* & & \sum e_i \otimes e^i \\ \downarrow & & \downarrow \\ V \otimes V^* & & \sum \rho(\tilde{\gamma}).e_i \otimes e^i \\ \downarrow & & \downarrow \\ \mathbb{C} & & \sum e^i(\rho(\tilde{\gamma}).e_i) \\ & & = tr(\rho(\tilde{\gamma})) \end{array}$$

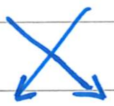
(CLASSICAL) SKEIN CATEGORY

RECALL THAT $\operatorname{Rep} G$ = (POSSIBLY INFINITE) DISCRETE SUM OF f.d. G-MODULE

LET $\mathcal{F} \subset \operatorname{Rep} G$ BE THE FULL SUBCATEGORY WHOSE OBJECTS ARE TENSOR PRODUCTS OF V AND V^*. (SOME SORT OF GENERATING SUBCATEGORY UNDER $\oplus$ + COMPLETION)

<u>MORPHISMS</u>:

V AND V^* ARE DUAL: HAVE 

 $V \otimes V^*$ $V^* \otimes V$

$$\operatorname{Hom}_G(V \otimes V^*, V) \cong \operatorname{Hom}_G(V, V \otimes V)$$

$\rightsquigarrow$ CAN DESCRIBE MORPHISMS IN TERMS OF V ONLY.

WE NEED TO DESCRIBE $\operatorname{Hom}(V^{\otimes k}, V^{\otimes l})$ $= 0$ if $k \neq l$

$$\mathbb{C}[\underset{\text{SYMMETRIC GROUP}}{S_k}] \xrightarrow{(*)} \operatorname{End}(V^{\otimes k})$$

SCHUR–WEYL DUALITY: This is surjective. IF $k \leq n$, This is also injective.

OBSERVE: $\Lambda^{n+1} V = 0$

KERNEL OF $(*)$ IF $k \geq n+1$ = 2-SIDED IDEAL GENERATED BY

$$\sum_{\sigma \in S_{n+1} \subset S_k} \varepsilon(\sigma)\, \sigma$$

$\varepsilon(\sigma)$: sign of σ

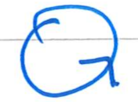
OTHER RELATION IN $\mathcal{F}$:  $= n = tr(id_V)$

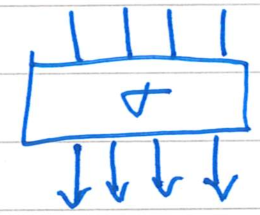
$\mathcal{T} \cong$ CATEGORY WITH

OBJECTS FINITE SEQUENCES OF $\{+, -\}$

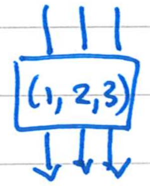
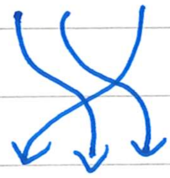
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\overset{\shortparallel}{V} \quad \overset{\shortparallel}{V^*}$

MORPHISMS = HOMOTOPY CLASSES OF "TANGLES" (ORIENTED) MODULO

 $= n$

$$\sum_{\sigma \in S_{n+1}} \varepsilon(\sigma)$$  $= 0$

SKEIN RELATION

EX:  $:=$ 

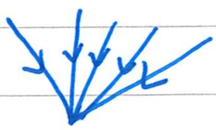
SKEIN PRESENTATION OF $\mathcal{T}$ / $Rep(G)$.

REM: $\Lambda^n V$ is 1-di — IT IS THE DETERMINANT REPR.

$\Lambda^n V \cong \mathbb{C}$ AS Sl_n-MODULE. (SEE PAPER [S;] IN REFS)

$$V^{\otimes n} \longrightarrow \mathbb{C}$$

$$v_1 \otimes \dots \otimes v_n \longmapsto \sum \varepsilon(\sigma)\, v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)} \in \Lambda^n V \cong \mathbb{C}$$

REPRESENT AS  ~~[illegible]~~ IN Sl_n-VERSION OF $\mathcal{T}$

IN PARTICULAR, FOR $n=2$

Sl_2: HAVE $V \otimes V \longrightarrow \mathbb{C}$ $\quad (tr(A) = tr(A^{-1}))$

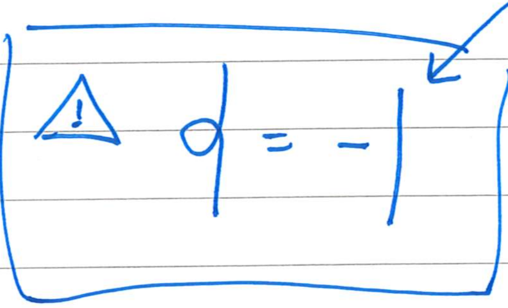
PAIRING ON V (NO NEED OF V^*!)

IF e_1, e_2 BASIS FOR V (SEE PAPER [Ti] "A SIGN THAT USED TO ANNOY ME...")

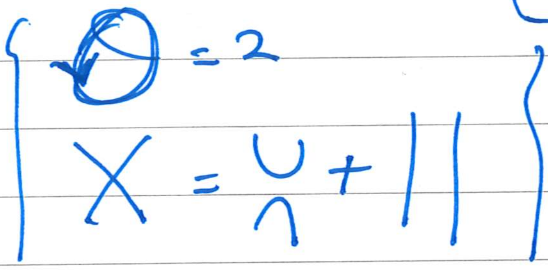
$(e_1, e_2) \longmapsto -1$ $\quad \cup = -\cup$

$\mathbb{C} \longrightarrow V \otimes V$

$1 \longmapsto e_1 \wedge e_2 \quad \rightsquigarrow \cap$



RELATION:



CLASSICAL KAUFFMANN SKEIN RELATIONS

LET $Sk(S)$ = CATEGORY WITH OBJECTS FINITE SEQUENCES OF POINTS IN S LABELLED $+$ OR $-$ AND MORPHISMS = HTPY CLASSES OF "FLAT" ORIENT TANGLES IN $S \times I$ MODULO SKEIN RELATIONS.

(NOTE: $\exists$ MAP $Br_n(S) \to$ "$S_n(S)$" $\longrightarrow End(\underbrace{+ \cdots +}_{n})$)

OBSERVATION: $\mathcal{O}(Ch(S)) = End_{Sk(S)}(\emptyset)$

$\rightsquigarrow$ GET A DESCRIPTION OF THE KERNEL OF $\mathcal{Y}_S \longrightarrow \mathcal{O}(Ch(S))$

BECAUSE $End_{Sk(S)}(\emptyset)$ ARE UNIONS OF LOOPS ON THE SURFACE $\rightsquigarrow$ RELATED TO THE MAP WE SAW BEFORE.

$\mathcal{O}(R(S))^G \rightsquigarrow$ G-EQUIVARIANT $\mathcal{O}(R(S))$-MODULE

REPLACE BY

$\left(\mathcal{O}(R(S))\text{-mod}\right)_{\mathrm{Rep}G} := QC(Ch(S))$

QUASI-COHERENT SHEAVES ON $Ch(S)$

$\in$

$V \in \mathrm{Rep}(G)$ s.t. V is $\mathcal{O}(R(S))$-MODULE

s.t. $\mathcal{O}(R(S)) \otimes V \longrightarrow V$ MORPHISM OF G-MODULES

$Ch(D^2)$ TRIVIAL BUT $Sk(D) \cong ∃$ NON-TRIVIAL

AND $QC(Ch(D)) \cong \mathrm{Rep}\,G$ BECAUSE

$R(D) = *$

$Ch(D) = */\!\!/G = BG$

SLOGAN OF THESE LECTURES:

$QC(Ch(S))$ IS BUILD BY GLUING TOGETHER COPIES OF $\mathrm{Rep}(G)$.

ANALOGOUS: $Sk(S) = \mathcal{O}(R(S))$-MODULES OF THE FORM $\mathcal{O}(R(S)) \otimes V^{\otimes n}$

(SEE PAPER ~~BY~~ [Co])

NOTE: LOOPS IN A GLUED SURFACE COME FROM TANGLES, NOT LOOPS, IN THE ORIGINAL SURFACES:

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