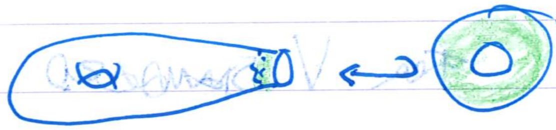
ADRIEN BROCHIER, LECTURE 2

IF $f : S \hookrightarrow S'$ BASEPOINT COMPATIBLE.

$\rightsquigarrow \pi_1(S, x) \longrightarrow \pi_1(S', x')$

BY RESTRICTION, GET

$$R(S', x') \longrightarrow R(S, x)$$
$$Ch(S') \longrightarrow Ch(S)$$

IN PARTICULAR, 

$$\rightsquigarrow \quad Ch(S) \longrightarrow Ch(Z)$$
$$R(S) \longrightarrow R(Z)$$

IS THE MAP μ FROM BEFORE.

1.2 COMBINATORIAL DESCRIPTION

WANT SOME DESCRIPTION OF $\mathcal{O}(G)$.
WILL GIVE THREE.

$$\mathcal{O}(G) = \mathbb{C}\left[(a_{ij})_{1 \leq i, j \leq n},\ \det(a_{ij})^{-1}\right]$$

("FRT PRESENTATION")

$\mathcal{O}(G)$ AND $U(\mathfrak{gl}_n)$ ARE DUAL TO EACH OTHER.
BOTH ARE HOPF ALGEBRAS

$$\mathcal{O}(G) \times \mathfrak{gl}_n \longrightarrow \mathbb{C}$$
$$(f, x) \longmapsto \frac{d}{dt} f(e^{tx})\Big|_{t=0}$$

"CANONICAL ELEMENT"

$X = \sum e_i \otimes e^i \in U(\mathfrak{gl}_n) \otimes \mathcal{O}(G)$ "

W f.d. G-MODULE. $\rightsquigarrow$ ALSO $\mathfrak{gl}_n$-MODULE

$X \rightsquigarrow X_W \in \mathrm{End}(W) \otimes \mathcal{O}(G)$ $\quad X_W = \sum \underbrace{e_i|_W}_{\in \mathrm{End}(W)} \otimes e^i$

W is CANONICALLY AN $\mathcal{O}(G)$-COMODULE

$W \longrightarrow W \otimes \mathcal{O}(G)$

X_V FOR V STANDARD n-dim REP

$X_V = \sum E_{ij} \otimes a_{ij}$, $a_{ij} \in \mathcal{O}(G)$

"ELEMENTARY MATRICES" CONSIDERED AS LIVING IN $\mathrm{End}(V) \cong \mathfrak{gl}_n$

Claim : THE a_{ij}'s ARE GENERATORS OF $\mathcal{O}(G)$

DEF : $X_V^1 = \sum E_{ij} \otimes id \otimes a_{ij}$ $\quad \in \mathfrak{gl}_n^{\otimes 2} \otimes \mathcal{O}(G)$

$X_V^2 = \sum id \otimes E_{ij} \otimes a_{ij}$

$\mathcal{O}(G)$ is GENERATED BY (a_{ij}) WITH RELATION

$X_V^1 X_V^2 = X_V^2 X_V^1$

$\det(a_{ij})^{-1}$

(REFORMULATES THE RELATIONS OF $\mathcal{O}(G)$

– WILL BE USEFUL LATER)

THAT THE a_{ij}'s COMMUTE

PETER-WEYL THEOREM

FOR ANY FINITE DIMENSIONAL G-MODULE W,
$\exists$ MAP $W^* \otimes W \longrightarrow \mathcal{O}(G)$

$\mathrm{End}(W) \cong$ $\quad f \otimes w \longmapsto [g \longmapsto f(g.w)] =: \varphi_{f,w}$

$\rightsquigarrow$ THM $\quad \mathcal{O}(G) \underset{\text{G-MODULE}}{\cong} \bigoplus_{\substack{\text{FIN. DIM} \\ \text{SIMPLE} \\ \text{G-MOD.}}} W^* \otimes W$

MULTIPLICATION: $\quad \varphi_{f,w} \cdot \varphi_{f',w'} = \varphi_{f \otimes f', w \otimes w'}$

$W \otimes Z \longrightarrow Z \otimes W$ IS A MAP OF
$w \otimes z \longmapsto z \otimes w$ G-MODULES

$$W \otimes Z \xrightarrow{\ \times\ } Z \otimes W$$

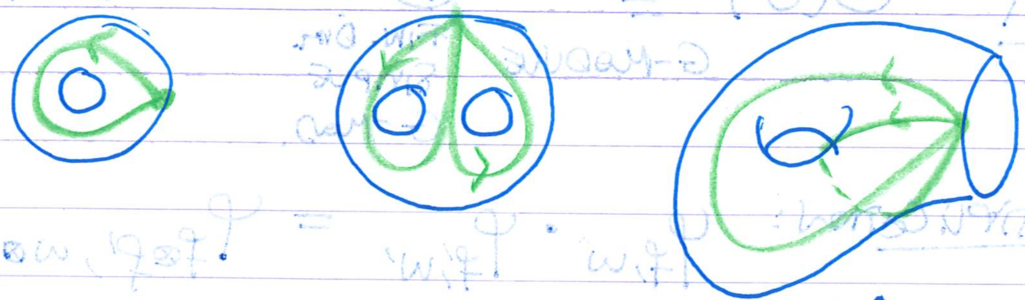
$$\begin{tikzcd}
\mathcal{O}(G) \otimes \mathcal{O}(G) & \\
W^* \otimes W \otimes Z^* \otimes Z \arrow[u] \arrow[d] & \text{MAP IN } \mathrm{Rep}(G) \\
Z^* \otimes W^* \otimes W \otimes Z & \\
(W \otimes Z)^* \otimes (W \otimes Z) \arrow[d] & \\
\mathcal{O}(G) &
\end{tikzcd}$$

$Z^* \otimes W^* \cong (W \otimes Z)^*$, $\quad W \otimes Z = W \otimes Z$

$\rightsquigarrow$ COMBINATORIAL DESCRIPTION OF $\mathcal{O}(G)$

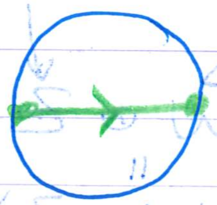
Def: A SKELETON on S is a GRAPH on S with VERTICES on the BOUNDARY and an ORDERING of the EDGES at EACH VERTEX, and an ORIENTATION of the EDGES, such that S DEFORMATION RETRACTS on the GRAPH.

Ex:



(FOR A VERTEX AT THE BOUNDARY, $\exists$ CANONICAL ORDERING OF THE HALF-EDGES GIVEN BY THE ORIENTATION OF S)

$\cong S$



THE CHOICE OF A SKELETON $\Gamma = (V, E) + \ldots$

$$R(S, V) \cong G^{|E|}$$

$\approx$ REPR. OF FUNDAMENTAL GROUPOID WITH BASEPOINTS V.

ACTION OF $G^{|V|} \curvearrowright G^{|E|}$: $\to\bullet$ G ACTS BY RIGHT-MULT

$\leftarrow\bullet$ —— LEFT-MULT

$\rightsquigarrow$: G ACTS BY CONJUGATION

$\rightsquigarrow$ ONLY ONE VERTEX IS AS BEFORE.

1.3) TOPOLOGICAL DESCRIPTION

FOR S ARBITRARY (OPEN OR CLOSED)
IF γ IS A FREE LOOP ON S,

- CHOOSE A BASEPOINT SOMEWHERE ON S
- CHOOSE A PATH FROM x TO A POINT ON γ

$\rightsquigarrow \tilde{\gamma} \in \pi_1(S, x)$

$\varphi_\gamma : Ch(S) \longrightarrow \mathbb{C}$ DEFINED BY

$[\rho] \in Ch(S)$, LIFT TO $\rho : \pi_1(S,x) \longrightarrow G$

$\varphi_\gamma([\rho]) := tr(\rho(\tilde{\gamma}))$ (DOES NOT DEPEND ON ALL THE CHOICES)

SEVERAL (POSSIBLY ZERO!) LOOPS ON S
$\rightsquigarrow$ PRODUCT OF THE CORRESPONDING FUNCTIONS.

DEFINE THE GOLDMAN ALGEBRA OF S:

$\mathcal{G}_S = \mathbb{C}$-LINEAR COMBINATION OF MULTI-LOOPS ON S / HTPY

PRODUCT = DISJOINT UNION
UNION = EMPTY LOOP.

"FIRST FUNDAMENTAL THM OF INVARIANT THEORY"

CLAIM: THERE IS A <u>SURJECTIVE</u> ALGEBRA MAP

$$\mathcal{G}_S \longrightarrow \mathcal{O}(Ch(S))$$

NOTE: $\mathcal{G}_S$ = POLYNOMIAL ALGEBRA ON FREE HTPY CLASSES OF LOOPS ON S.

SKEIN THEORY: WHAT IS THE KERNEL OF THE ABOVE MAP? ___ FOR NEXT TIME.