# ADRIEN BROCHIER : QUANTUM CHARACTER VARIETIES AND TOPOLOGICAL FIELD THEORIES

## LECTURE 1

http://abrochier.org/mc WITH TENTATIVE PLAN + LINKS TO RELEVANT ARTICLES

## 0. OUTLINE

LET $V$ BE AN $n$-di VECT SPACE / $\mathbb{C}$

$G = GL(V)$ (OR $SL(V)$)

LET $S$ BE A TOPOLOGICAL SURFACE (COMPACT, ORIENTED, MAY HAVE BOUNDARY)

CHARACTER VARIETY OF $S$:

$$Ch(S) = \{n\text{-di LINEAR REPR. OF } \pi_1(S)\}/_{\simeq}$$

$$= \{\rho : \pi_1(S) \to G\}/G^{ad}$$

?

NEED TO MAKE THIS PRECISE

[ATIYAH-BOTT, GOLDMAN]: POISSON STRUCTURE.

(AT LEAST) TWO DIFFERENT DESCRIPTIONS

COMBINATORIAL ($\sim$ GENERATORS FOR $\pi_1(S)$ / PRESENTATION OF $S$)

TOPOLOGICAL (FREE LOOPS ON $S$)

$\Big\}$ QUANTIZATION
$\downarrow$

| | |
|---|---|
| QUANTUM ALGEBRA | SKEIN THEORY |
| (USING QUANTUM GROUPS) | (LOOPS $\rightsquigarrow$ KNOTS) |

MAIN GOAL OF THE LECTURES: REFORMULATE AND GENERALIZE ALL OF THAT IN THE LANGUAGE OF TOPOLOGICAL FIELD THEORIES.

$\rightsquigarrow$ WANT A CANONICAL, TOPOLOGICAL CONSTRUCTION, COMPATIBLE WITH GLUING SURFACES AND EASY TO COMPUTE.

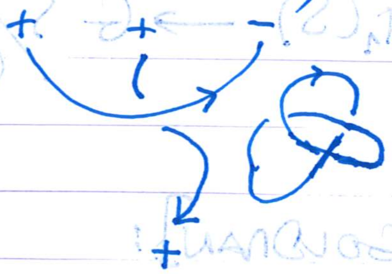
<u>MAIN IDEA</u>: WORK WITH A CATEGORY OF SHEAVES ON $Ch(S)$.

<u>ANALOGY</u> (TO BE MADE PRECISE)

IF YOU CARE ABOUT THE SPACE OF LINKS IN $\mathbb{R}^3$, IT'S BETTER TO REPLACE IT WITH THE CATEGORY OF TANGLES:

OBJ = FINITE SEQUENCES OF $\{+,-\}$

MORPH = (FRAMED, OR.) TANGLES




$End(\emptyset)$ = SPACE OF LINKS.

CATEGORY DESCRIBABLE IN TERMS OF GENERATORS AND RELATIONS. EASIER TO DEFINE A FUNCTOR FROM THAT CATEGORY TO DEFINE A LINK INVARIANT.

SPACE OF LINKS $\rightsquigarrow$ FUNCTIONS ON $Ch(S)$

CATEGORY OF TANGLES $\rightsquigarrow$ CATEGORY OF SHEAVES ON $Ch(S)$.

## 1) CHARACTER VARIETIES

### 1.1. DEFINITION

LET $\Gamma$ BE A DISCRETE GROUP, (FINITELY GEN. PRESENTED)

**Def**: THE REPRESENTATION VARIETY OF $\Gamma$ IS

$$R(\Gamma) = \{ \rho : \Gamma \longrightarrow G \}$$

$G = GL(V)$ or $SL(V)$

CHOOSE GENERATORS AND RELATIONS FOR $\Gamma$:

$$\Gamma = \langle x_1, \dots, x_k \mid R_1, \dots, R_\ell \rangle$$

$$\rightsquigarrow R(\Gamma) \subset G^k$$

$$\shortparallel$$

$$\{ (X_1, \dots, X_k) \in G^k \mid \underbrace{R_i(X_1, \dots, X_k) = Id}_{} \}$$

POLYNOMIAL EQUATIONS IN THE COEFFICIENTS OF THE $X_i$'S

$R(\Gamma)$ IS A VARIETY

$\rightsquigarrow$ VARIETY

(AFFINE ALGEBRAIC VARIETY)

**PROP**: FOR ANY TWO PRESENTATIONS OF $\Gamma$, THE CORRESPONDING ALG. VARIETY ARE CANONICALLY ISOMORPHIC.

IN PARTICULAR, $x \in S$

$$R(S,x) = R(\pi_1(S,x))$$

(DOES NOT DEPEND ON $x$, BUT NON-CANONICALLY)

IF $G \circlearrowright X$ AFFINE ALG. VARIETY

INSTEAD OF TAKING THE QUOTIENT $X/G$, WE SAY WHAT THE FUNCTIONS ON THE QUOTIENT SHOULD BE:

Def: THE AFFINE QUOTIENT $X/\!/G$ IS DEFINED BY $\mathcal{O}(X/\!/G) = \mathcal{O}(X)^G$ = INVARIANT FUNCTIONS ON $X$.

($m : \mathcal{O}(X) \otimes \mathcal{O}(X) \longrightarrow \mathcal{O}(X)$ IS A MAP OF $G$-MODULES,
$\mathcal{O}(X)$ IS AN ALGEBRA IN THE MONOIDAL CATEGORY Rep $G$)

$\rightsquigarrow$ TAKING INVARIANTS GIVES AN ALGEBRA WHICH IS NOT IN GENERAL THE ALG OF FUNCTIONS ON THE QUOTIENT.

Ex: $G \circlearrowright \mathrm{End}(V)$ BY CONJUGATION.

$$\mathrm{End}(V)/\!/G^{ad} = \text{DIAGONAL MATRICES}/S_n \quad S_n = \text{SYM. GROUP}$$

(NON-DIAGONAL MATRICES ARE IN THE CLOSURE OF THE ORBIT OF DIAGONAL MATRICES)

$$Ch(\Gamma) = Rep(\Gamma)/\!/G = R(\Gamma)/\!/G$$

$Ch(\mathbb{Z})$ = INVERTIBLE DIAGONAL MATRICES $/S_n$

= SEMI-SIMPLE REPRESENTATIONS OF $\mathbb{Z}$ / EQUIV.

THM ([LM])

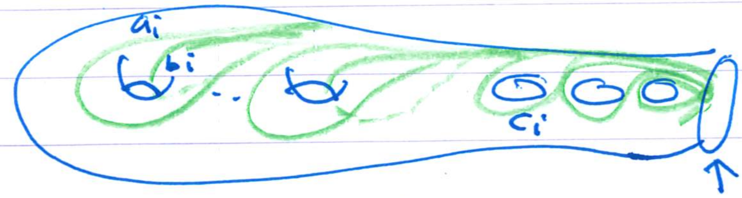
FOR ANY $\Gamma$, THERE ARE BIJECTIONS BETWEEN

- POINTS OF $Ch(\Gamma)$
- EQUIV. CLASSES OF SS G-REPR OF $\Gamma$
- SET OF (n-di) CHARACT. OF $\Gamma$

$$\rho : \Gamma \longrightarrow G$$
$$\chi_\rho : x \longmapsto tr(\rho(x))$$

IF $\Gamma = \pi_1(S, x)$, $Ch(S) := Ch(\pi_1(S,x))$ IS CANONICALLY INDEP. OF $x$.

IF $S = S_{g,n}$, $n > 0$. $\pi_1(S) = F^{2g+n-1}$ IS A FREE GROUP.




DISTINGUISHED BDRY

IF $n=0$, $\pi_1(S) = \langle a_i, b_i \mid \prod_i [a_i, b_i] = 1 \rangle$

COMMUTATOR

IF $\mathring{S} = S \setminus D$, TAKING A LOOP AROUND THE UNIQUE BOUNDARY COMPONENT GIVES

$$\mu : R(\pi_1(\mathring{S}, x)) \longrightarrow R(\underset{\substack{\| \\ S^1 \times I}}{\circledcirc}) = G$$

$$\{A_i, B_i\} \longmapsto \prod_i [A_i, B_i]$$

$$\rightsquigarrow \underset{\substack{\uparrow \\ \text{CLOSED SURFACE}}}{Ch(S)} = \mu^{-1}(\{Id\})/G$$

THE BDRY GENERATOR GOES TO THE IDENTITY.