Helly graphs and groups

Masterclass "Topics in Geometric Group Theory"

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Københavns Universitet

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Finite-type Artin groups

Definition (Artin group)

A finite simplicial graph Γ with edges labelled by $\{2, 3, 4, \ldots\}$ defines a presentation of the *Artin group* A_Γ:

$$A_\Gamma = \langle a \in V(\Gamma) \mid \underbrace{aba\cdots}_{m} = \underbrace{bab\cdots}_{m} \text{ for each edge } ab \text{ labelled with } m \rangle$$

Definition

An Artin group A_Γ is of *finite type* if the Coxeter group C_Γ is finite.

Theorem

Finite-type Artin groups are Helly.

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FC-type Artin groups are Helly.

Garside groups

Theorem

Weak Garside groups of finite type are Helly.

C(4)-T(4) small cancellation groups

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Finitely presented C(4)-T(4) (graphical) small cancellation groups are Helly.

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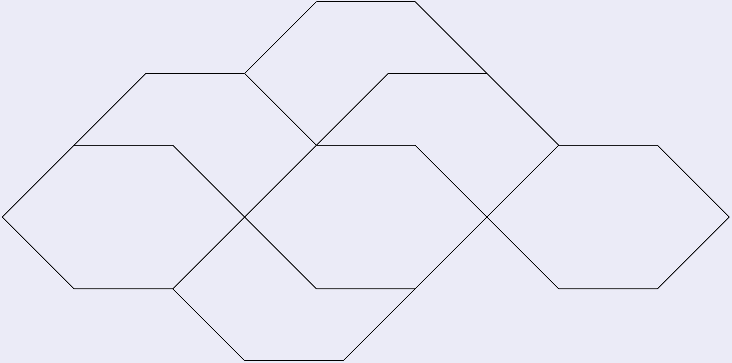
Proof.

Consider a ‘thickening’ of the Cayley complex:



Buildings

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Uniform lattices in Euclidean buildings of type C are Helly.

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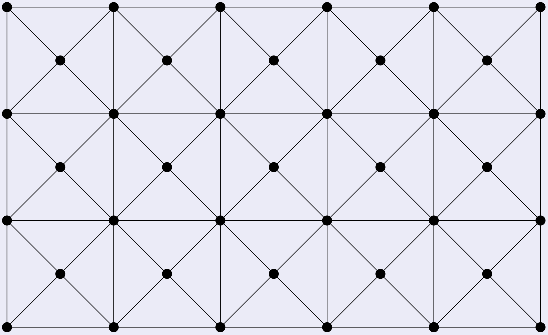

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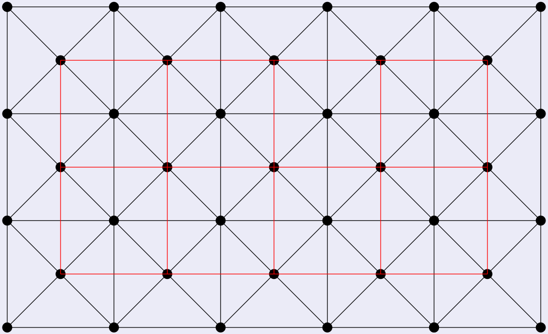

Properties of Helly groups

Theorem (Properties of Helly groups)

Let G be a group acting geometrically on a Helly graph Γ. Then:

1. *The clique complex $X(\Gamma)$ of Γ is a finite-dimensional cocompact model for the classifying space $\underline{E}G$ for proper actions. As a particular case, G is always of type F_∞ and of type F when it is torsion-free.*

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4. *G has at most quadratic Dehn function.*

Constructions preserving Hellyness

Theorem

Let $G, G_1, G_2, \ldots, G_n$ be Helly groups. Then:

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Proof.

"Amalgamation of Helly graphs along a vertex is Helly. Strong product of Helly graphs is Helly. Fixed-point set is Helly." □

Hellyfication

Theorem (Jawhari-Pouzet-Misane, Pesch)

For every graph Γ *there exists a minimal Helly graph* $Helly(\Gamma)$*, called Hellyfication of* Γ *into which* Γ *embeds isometrically.*

Proof.

Consider the space $\mathbb{Z}^\Gamma$ of integer-valued functions with the supremum metric $d(f,g) = \sup_{x\in\Gamma} |f(x) - g(x)|$.

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Additionaly, $\mathrm{Helly}(\Gamma) = E(\Gamma) \cap \mathbb{Z}^\Gamma$.

□

Injective hull vs Hellyfication

Theorem

Let Γ be a locally finite Helly graph.

1. *The injective hull $E(\Gamma)$ of Γ is proper and has the structure of a locally finite polyhedral complex with only finitely many isometry types of n-cells, isometric to injective polytopes in $(\mathbb{R}^n, d_\infty)$, for every $n \geq 1$. Moreover, $d_H(E(\Gamma), e(\Gamma)) \leq 1$. Furthermore, if Γ has uniformly bounded degrees, then $E(\Gamma)$ has finite combinatorial dimension.*

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Helly groups act geometrically on spaces with convex, reversible, consistent geodesic bicombing = act geometrically on CAT(0)-like spaces

Further properties of Helly groups

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4. *The asymptotic cones of G are contractible.*

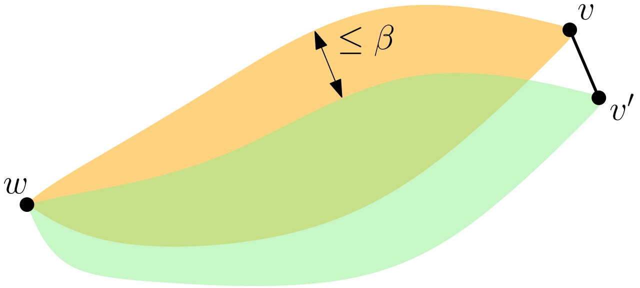
Proof.

Follows immediately from results by Descombes-Lang, Kasprowski-Rüping, Fukaya-Oguni. □

β-stable intervals

Definition (Lang)

For $\beta \geq 1$, the graph Γ has *β-stable intervals* if for every triple of vertices w, v, v' with $v \sim v'$, we have $d_H(I(w,v), I(w,v')) \leq \beta$, where d_H denotes the Hausdorff distance.



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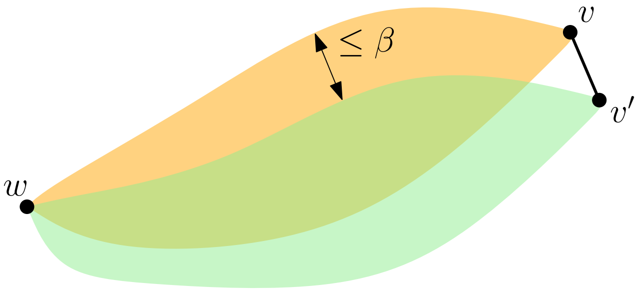

Remark

This property is equivalent to the FFTP.

Graphs with β-stable intervals

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Let Γ be a locally finite graph with β-stable intervals. Then the injective hull of Γ is proper (that is, bounded closed subsets are compact) and has the structure of a locally finite polyhedral complex with only finitely many isometry types of n-cells, isometric to injective polytopes in $(\mathbb{R}^n, d_\infty)$, for every $n \geq 1$.

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Weakly modular graphs (in particular, Helly graphs) have 1-stable intervals.

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Example

For Γ being the 1-skeleton of a regular cubical grid in $\mathbb{E}^3$ or a regular triangulation of $\mathbb{E}^2$ we have $d_H(e(\Gamma), E(\Gamma)) = \infty$, equivalently, $d_H(e(\Gamma), \mathrm{Helly}(\Gamma)) = \infty$.

Coarse Helly

Definition

A metric space (X, d) has the *coarse Helly property* if there exists $\delta \geq 0$ such that for any family $\{B_{r_i}(x_i) : i \in I\}$ of pairwise intersecting closed balls of X, the intersection $\bigcap_{i \in I} B_{r_i+\delta}(x_i)$ is not empty.

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$f(x)+f(y) > d(x, z)+\delta+\epsilon+d(y, z)-\delta = d(x, z)+d(y, z)+\epsilon \geq d(x, y)+\epsilon$, a contradiction. End of Lecture 2 □