Helly graphs and groups

Masterclass “Topics in Geometric Group Theory”

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Københavns Universitet

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Dismantlability

Definition (Dismantlability)

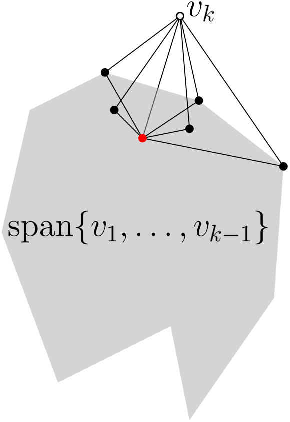
A finite graph Γ is *dismantlable* if its vertices can be enumerated as $v_1, v_2, v_3, \ldots., v_n$ such that for every $1 < k \leqslant n$ the vertex v_k is *dominated* in the subgraph induced by $v_1, \ldots, v_k$.

Theorem

Balls in locally finite Helly graphs are dismantlable.

Corollary

The clique complex $X(\Gamma)$ of a locally finite Helly graph Γ is contractible. Finite groups acting on such Helly graphs fix cliques. Fixed point sets are contractible.



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5. Γ *is weakly modular* 1*-Helly graph*

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At each step we show that B_i satisfies the triangle condition and the enhanced quadrangle condition with respect to O, as well as, every ball is dismantlable.
It follows that $\bigcup B_i$ is simply connected. By universal properties of universal coverings the construction does not depend of the choice of O, so the 1-skeleton of $\widetilde{X}$ is Helly $\square$

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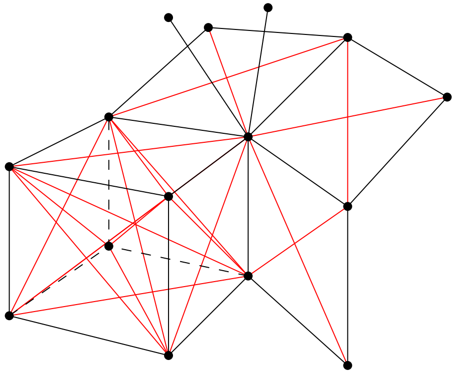
Example

Cocompact $\mathrm{CAT}(0)$ cubical groups are Helly.
The proof goes via convexity of balls or via the local-to-global characterization.



Finite-type Artin groups

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A finite simplicial graph Γ with edges labelled by $\{2, 3, 4, \ldots\}$ defines a presentation of the *Artin group* A_Γ:

$$A_\Gamma = \langle a \in V(\Gamma) \mid \underbrace{aba\cdots}_{m} = \underbrace{bab\cdots}_{m} \text{ for each edge } ab \text{ labelled with } m \rangle$$

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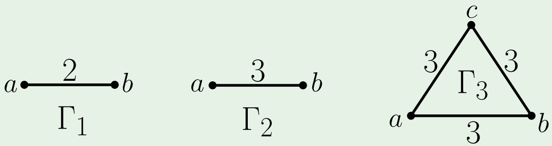

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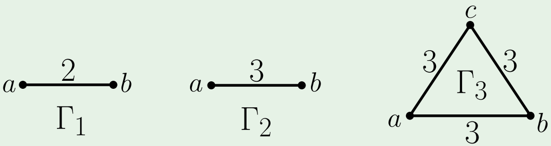

$$A_{\Gamma_1} = \langle a, b \mid ab = ba \rangle \cong \mathbb{Z}^2; \; A_{\Gamma_2} = \langle a, b \mid aba = bab \rangle$$
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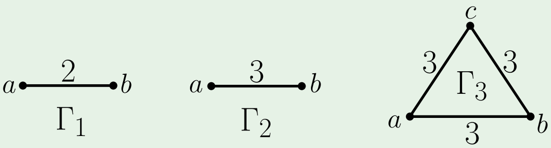

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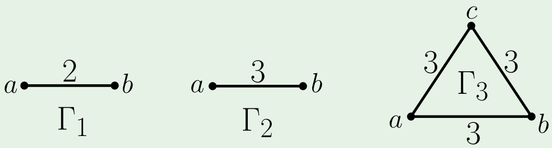

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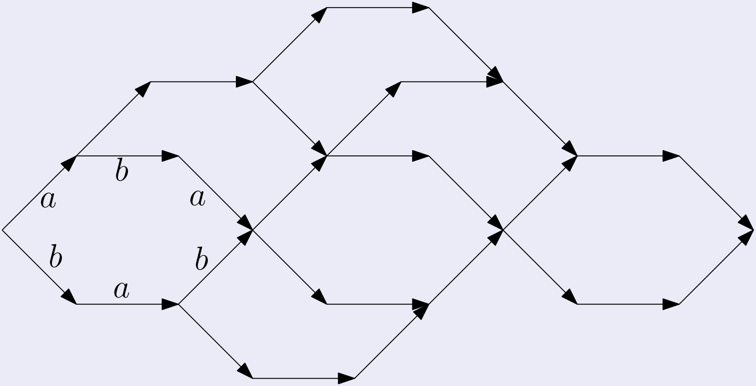

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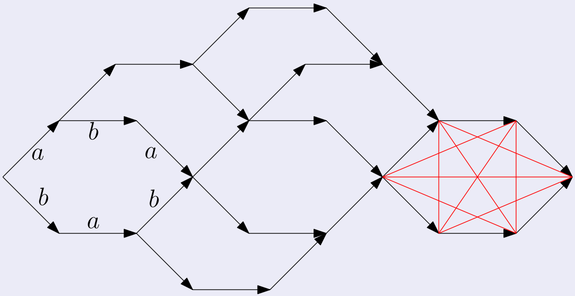

End of Lecture 2

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