Helly graphs and groups

Masterclass "Topics in Geometric Group Theory"

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Geodesic bicombing

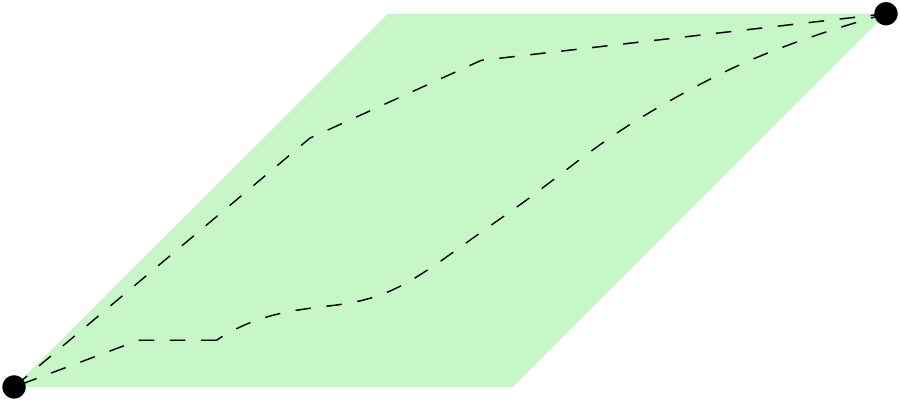
Definition (Geodesic bicombing)

A *geodesic bicombing* on a metric space (X, d) is a map

$$\sigma\colon X \times X \times [0,1] \to X,$$

such that for every pair $(x, y) \in X \times X$ the function $\sigma_{xy} := \sigma(x, y, \cdot)$ is a constant speed geodesic from x to y.
We call σ *convex* if the function $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is convex for all $x, y, x', y' \in X$.
The bicombing σ is *consistent* if $\sigma_{pq}(\lambda) = \sigma_{xy}((1-\lambda)s + \lambda t)$, for all $x, y \in X$, $0 \leq s \leq t \leq 1$, $p := \sigma_{xy}(s)$, $q := \sigma_{xy}(t)$, and $\lambda \in [0,1]$.
It is called *reversible* if $\sigma_{xy}(t) = \sigma_{yx}(1-t)$ for all $x, y \in X$ and $t \in [0,1]$.

Bicombing in $(\mathbb{R}^2, d_\infty)$



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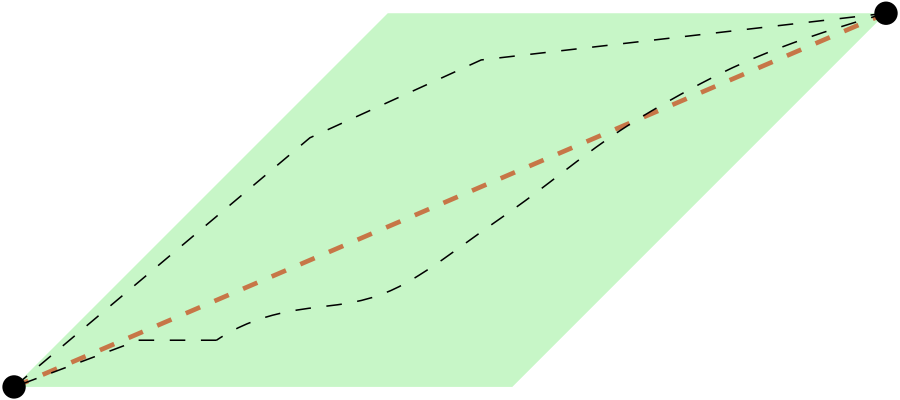

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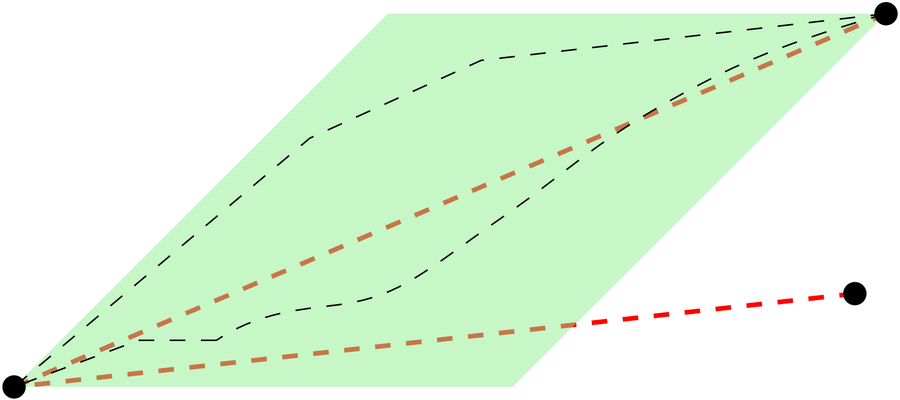

Bicombing in injective spaces

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A proper injective metric space X of finite combinatorial dimension admits a unique convex, consistent, reversible geodesic bicombing.

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Remark

In particular, the bicombing above is invariant under automorphisms.

Properties of injective metric spaces

1. contractibility
2. fixed point properties for finite group actions
3. classification of isometries
4. Flat Torus theorem [Descombes-Lang]
5. characterization of hyperbolicity via non-existence of flats

...

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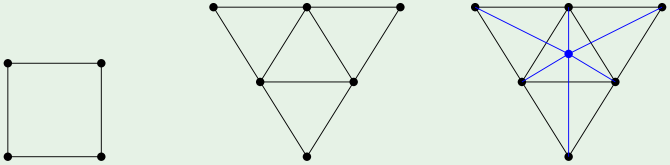
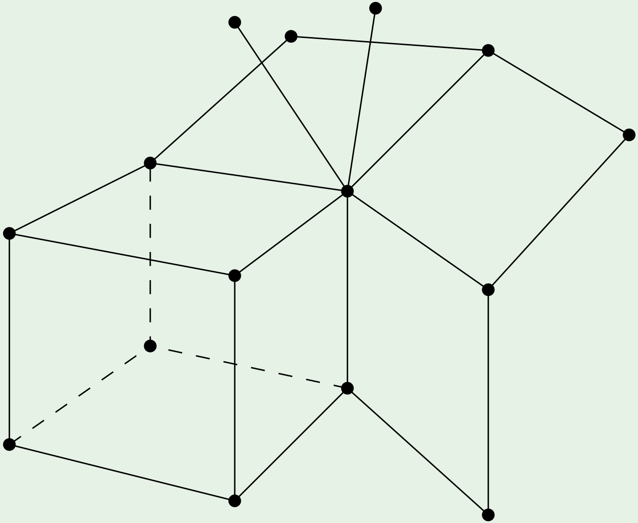
Example



clique-Helly not Helly — not clique-Helly — Helly

Helly graphs - example

Example (Thickening of a $\mathrm{CAT}(0)$ cube complex)



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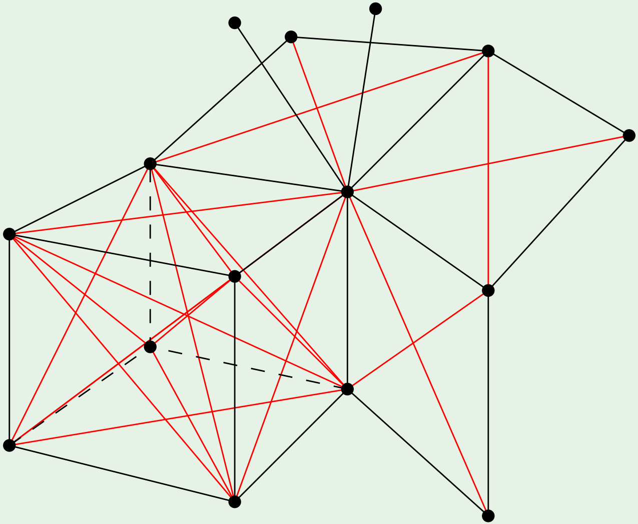

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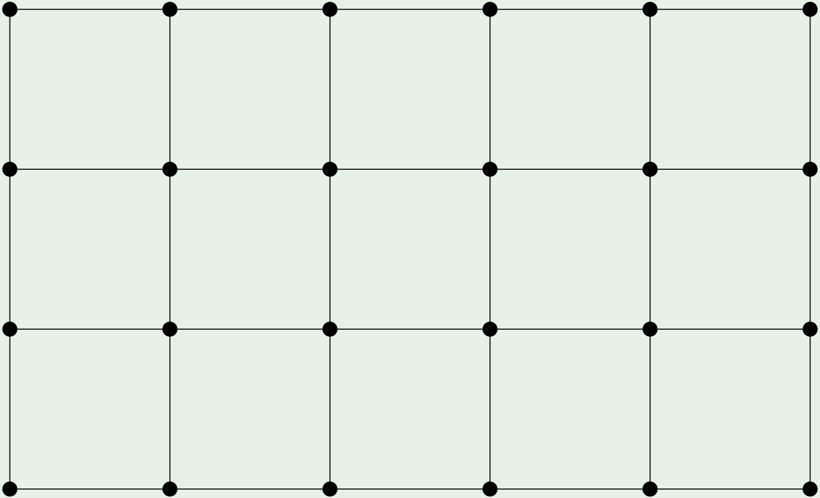

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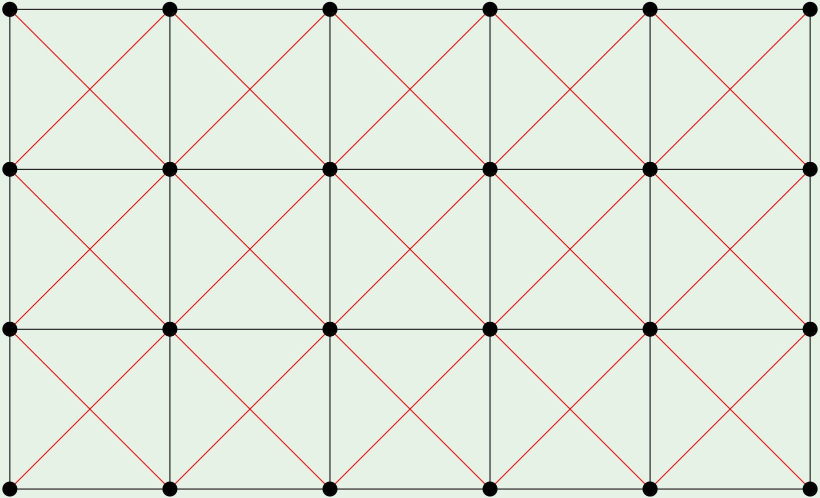

Helly graph

Theorem

Helly graphs are weakly modular. Moreover, they satisfy a stronger version of the quadrangle condition:

- *if there exists* $z \sim v, w$ *with* $d(u, z) = n + 1$ *then there exists* $x \sim v, w$ *with* $d(u, x) = n - 1$, *and* $x' \sim z, v, w, x$

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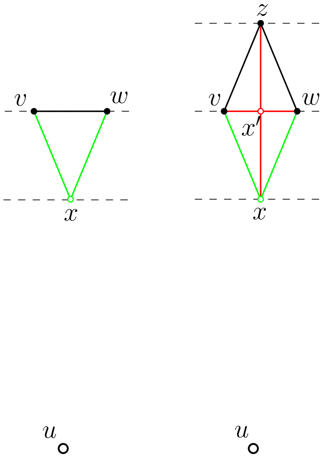

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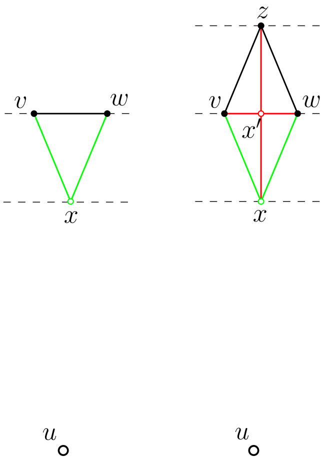

Corollary

The triangle complex of a Helly graph is simply connected. The isoperimetric function is at most quadratic.

Dismantlability

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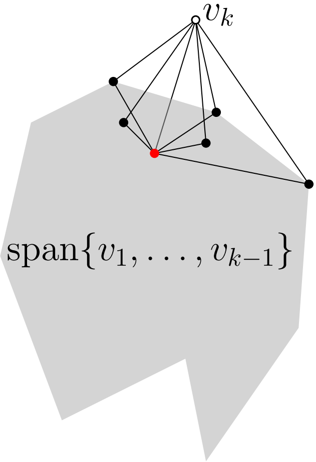
A finite graph Γ is *dismantlable* if its vertices can be enumerated as $v_1, v_2, v_3, ..., v_n$ such that for every $1 < k \leqslant n$ the vertex v_k is *dominated* in the subgraph induced by $v_1, \ldots, v_k$.



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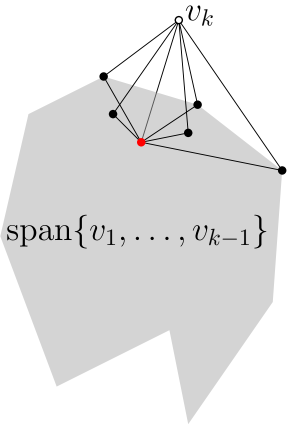
Corollary

The clique complex $X(\Gamma)$ of a locally finite Helly graph Γ is contractible. Finite groups acting on such Helly graphs fix cliques. Fixed point sets are contractible.



End of Lecture 2