Helly graphs and groups
Masterclass “Topics in Geometric Group Theory”

Damian Osajda

Københavns Universitet

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Sources


Sources

Victor Chepoi’s course notes for 2019 Simons Semester in Warsaw, available at:

Thomas Heattel’s course notes for 2023 CRM Semester in Montreal, available at:
arXiv:2307.00414
Sources


Sources


Thomas Haettel, Nima Hoda, Harry Petyt, **Coarse injectivity, hierarchical hyperbolicity and semihyperbolicity**, *Geom. Topol.* 27 (2023), 1587-1633.


Examples of groups acting geometrically on Helly graphs:
(Gromov) hyperbolic groups, (cocompact) $\text{CAT}(0)$ cubical groups,
uniform lattices in many Euclidean buildings, FC-type Artin groups,
finite-type Garside groups, fin. pres. graphical $\text{C}(4)$-$\text{T}(4)$ small cancellation
groups, ...
Helly groups

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Group theoretic constructions preserving Hellyness:
direct product, graph product, free product (and HNN extension) with amalgamation over finite subgroups, some graphs of groups, relative hyperbolicity, quotient by finite normal subgroup, ...
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Group theoretic constructions preserving Hellyness:
direct product, graph product, free product (and HNN extension) with amalgamation over finite subgroups, some graphs of groups, relative hyperbolicity, quotient by finite normal subgroup, . . .

Properties of Helly groups:
biautomaticity, finiteness properties, finitely many conjugacy classes of finite subgroups, Farrell-Jones conjecture, coarse Baum-Connes conjecture, EZ-boundary, . . .
Outline of the course:

1. Basics of Geometric Group Theory
2. Helly property, injective metric spaces, Helly graphs
3. Features of Helly graphs
4. Helly groups: examples and properties
5. Further topics
We will consider simplicial graphs, that is, undirected graphs without loops and multiple edges.
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Graphs

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A connected graph $\Gamma$ will be treated as a metric space $(V(\Gamma), d)$ where $d$ is the path metric.

A *tree* is a connected graph without cycles.
Definition (Group action)

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A trivial action $G \to \{1\}$. 
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An *action by automorphisms* on a graph $X$, when $\text{Aut}(X)$ is the group of (simplicial) automorphisms of $X$. 
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**Motto:** In Geometric Group Theory we study groups via their actions on spaces equipped with some geometry.
...so better the actions be nice.
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An action of $G$ on a metric space $X$ is \textit{proper}, if for every compact $K \subseteq X$, the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite.

Example 1
The action of $\mathbb{Z}$ on $\mathbb{R}$ by translations is geometric, moreover free (stabilizers are trivial).

Example 2
The action of $\mathbb{Z}$ on $\mathbb{R}^2$ by translations along one coordinate is proper, even free, but not cocompact.

Example 3
The action of $\mathbb{Z}^2$ on $\mathbb{R}$ via $(g, h) x = g + x$ is cocompact but not proper.
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Motto II: A group $G$ and a space acted geometrically upon $G$ look “alike”.
Definition (Cayley graph)

For a group $G$ and its generating set $S$, the Cayley graph $\text{Cay}(G, S)$ has the vertex set $G$ and edges of the form $\{g, gs\}$, for $g \in G$ and $s \in S$. 
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Example

$\text{Cay}(\mathbb{Z}, \{1\})$

A group acts geometrically, and freely by automorphisms (preserving types of edges) on its Cayley graph via $h \cdot g = hg$. 
Cayley graph

Example

$\text{Cay}(\mathbb{Z}^2, \{(0, 1), (1, 0)\})$
Cayley graph

Example

Free group: Cay\( (F(a, b), \{a, b\}) \)
Gromov hyperbolicity

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**Example**

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Example

- a bounded space;
- a line;

$\text{H}_n$, $X \times Y$, for a hyperbolic space $X$ and a bounded $Y$. 

$\text{E}_2$ is not hyperbolic.
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Example

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Example
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Weakly modular graph

Definition (Triangle condition)
A (simplicial) graph satisfies the triangle condition at distance $k > 0$ with respect to a vertex $u$, denoted $\text{TC}(u, k)$, if for any two vertices $v, w$ with $1 = d(v, w) < d(u, v) = d(u, w) = k$ there exists a common neighbor $x$ of $v$ and $w$ such that $d(u, x) = k - 1$. 

$\text{TC}(u, k)$
Definition (Quadrangle condition)

A (simplicial) graph satisfies the *quadrangle condition at distance* \( k > 0 \) *with respect to a vertex* \( u \), denoted \( \text{QC}(u, k) \), if for any three vertices \( v, w, z \) with \( d(v, z) = d(w, z) = 1 \) and \( 2 = d(v, w) \leq d(u, v) = d(u, w) = d(u, z) - 1 = k \), there exists a common neighbor \( x \) of \( v \) and \( w \) such that \( d(u, x) = k - 1 \).
Weakly modular graph

Definition (Weakly modular graph)

A (simplicial) graph is weakly modular if it satisfies conditions TC\((u, k)\) and QC\((u, k)\) for all \(u\) and \(k\).

\[\begin{align*}
\text{TC}(u, k) & \quad \iff \\
\text{QC}(u, k) & \quad \iff \\
\end{align*}\]
Definition (Median graph)

An interval $I(v, w)$ between vertices $v, w$ in a graph $\Gamma$ is the set of all vertices on geodesics from $v$ to $w$ (that is, $u$ such that $d(v, w) = d(v, u) + d(u, w)$).
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A graph is *median* (aka 1-skeleton of a $\text{CAT}(0)$ cubical complex) if for any vertices $u, v, w$ the intersection $I(u, v) \cap I(v, w) \cap I(w, u)$ is a single vertex (called the *median of $u, v, w$*).
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Example

Tree
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Example
Tree
1-skeleton of the standard cubulation of $\mathbb{E}^n$
Definition (Helly property)

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**Example (Helly families)**

1. axis-parallel boxes in $\mathbb{R}^n$
2. finite subtrees of a tree
3. a finite family of half-spaces of a CAT(0) cube complex.
Helly property - examples

Example (Gated subsets)
A subset $Y$ of a metric space $(X, d)$ is \textit{gated} if for every point $x \in X$ there exists a vertex $x' \in Y$, called the \textit{gate} of $x$, such that $x' \in I(x, y)$, for every $y \in Y$. A finite family of gated subsets has the Helly property.

Example (Intervals in lattices)
A \textit{lattice} is a poset $(P, \leq)$ with g.l.b. (called \textit{meet}) and l.u.b. (\textit{join}) for each pair of elements. An \textit{interval} in a lattice is a subset of the form $\{x | a \leq x \leq b\}$. A finite family of intervals in a lattice has the Helly property.
Injective metric spaces

Let \((X, d)\) be a geodesic metric space

**Definition (Injective space)**

\(X\) is injective if the family of balls has the Helly property.
Injective metric spaces

Let $(X, d)$ be a geodesic metric space

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**Example**

1. $(\mathbb{R}^n, d_{\infty})$
2. an $\mathbb{R}$-tree
3. $(\mathbb{R}^2, d_2)$ is not injective!
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**Remark**

Actually, the definition of an injective space above is not so proper...
Injective spaces

**Theorem (Characterizations of injectivity)**

Let \((X, d)\) be a geodesic metric space. TFAE:

1. \(X\) is injective
2. \(X\) is hyperconvex
3. [Aronszajn-Panitchpakdi, 1956] \((Y, X)\) has the **extension property**, for every metric space \(Y\) (for the category of metric spaces with 1-Lipschitz maps)
4. \(X\) is an absolute retract (for the category of metric spaces with 1-Lipschitz maps)
Definition (Injective hull)

An *injective hull* (or *tight span*, or *injective envelope*, or *hyperconvex hull*) of \((X, d)\) is a pair \((e, E(X))\) where \(e : X \to E(X)\) is an isometric embedding into an injective metric space \(E(X)\), and such that no injective proper subspace of \(E(X)\) contains \(e(X)\). Two injective hulls \(e : X \to E(X)\) and \(f : X \to E'(X)\) are *equivalent* if they are related by an isometry \(i : E(X) \to E'(X)\).
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Theorem (Isbell 1964)

Every metric space \((X, d)\) has an injective hull and all its injective hulls are equivalent.
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Theorem (Isbell 1964)

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Remark

*Injective hulls were rediscovered by Dress in 1984, Chrobak-Larmore in 1994...*
Isbell’s construction

Let \((X, d)\) be a metric space. Consider the space \(\mathbb{R}^X\) of real-valued functions with the supremum metric \(d(f, g) = \sup_{x \in X} |f(x) - g(x)|\).
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\[d(f, g) = \sup_{x \in X} |f(x) - g(x)|.\]
The Kuratowski embedding \(e : X \to \mathbb{R}^X : x \mapsto d(x, \cdot)\) is an isometric embedding.

A function \(f \in \mathbb{R}^X\) is called a metric form if 
\[f(x) + f(y) \geq d(x, y).\]
It is extremal if it is point-wise minimal.

We define \(E(X)\) as the space of extremal metric forms. 
\((e, E(X))\) is the injective hull of \(X\).
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Injective hulls - examples
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\[(0, a + b, c + a)\]

\[(a, b, c)\]

\[(a + b, 0, b + c)\]

\[(c + a, b + c, 0)\]
Injective hulls - examples

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Definition (Geodesic bicombing)

A \textit{geodesic bicombing} on a metric space \((X, d)\) is a map

\[ \sigma : X \times X \times [0, 1] \to X, \]

such that for every pair \((x, y) \in X \times X\) the function \(\sigma_{xy} := \sigma(x, y, \cdot)\) is a constant speed geodesic from \(x\) to \(y\).
Geodesic bicombing

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We call \(\sigma\) **convex** if the function \(t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))\) is convex for all \(x, y, x', y' \in X\).
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The bicombing \(\sigma\) is consistent if \(\sigma_{pq}(\lambda) = \sigma_{xy}((1 - \lambda)s + \lambda t)\), for all \(x, y \in X, 0 \leq s \leq t \leq 1, p := \sigma_{xy}(s), q := \sigma_{xy}(t),\) and \(\lambda \in [0, 1]\).
A *geodesic bicombing* on a metric space $(X, d)$ is a map
\[
\sigma : X \times X \times [0, 1] \to X,
\]
such that for every pair $(x, y) \in X \times X$ the function $\sigma_{xy} := \sigma(x, y, \cdot)$ is a constant speed geodesic from $x$ to $y$.

We call $\sigma$ **convex** if the function $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is convex for all $x, y, x', y' \in X$.

The bicombing $\sigma$ is **consistent** if $\sigma_{pq}(\lambda) = \sigma_{xy}((1 - \lambda)s + \lambda t)$, for all $x, y \in X$, $0 \leq s \leq t \leq 1$, $p := \sigma_{xy}(s)$, $q := \sigma_{xy}(t)$, and $\lambda \in [0, 1]$.

It is called **reversible** if $\sigma_{xy}(t) = \sigma_{yx}(1 - t)$ for all $x, y \in X$ and $t \in [0, 1]$. 

End of Lecture 1