Helly graphs and groups

Masterclass “Topics in Geometric Group Theory”

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Københavns Universitet

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Sources

Jérémie Chalopin, Victor Chepoi, Hiroshi Hirai, Damian Osajda, **Weakly modular graphs and nonpositive curvature**, *Mem. Amer. Math. Soc.* 268 (2020), no. 1309.

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Helly groups

Examples of groups acting geometrically on Helly graphs:
(Gromov) hyperbolic groups, (cocompact) $\mathrm{CAT}(0)$ cubical groups, uniform lattices in many Euclidean buildings, FC-type Artin groups, finite-type Garside groups, fin. pres. graphical C(4)-T(4) small cancellation groups,...

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Group theoretic constructions preserving Hellyness:
direct product, graph product, free product (and HNN extension) with amalgamation over finite subgroups, some graphs of groups, relative hyperbolicity, quotient by finite normal subgroup, ...

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Properties of Helly groups:
biautomaticity, finiteness properties, finitely many conjugacy classes of finite subgroups, Farrell-Jones conjecture, coarse Baum-Connes conjecture, EZ-boundary, ...

Outline of the course:

1. Basics of Geometric Group Theory
2. Helly property, injective metric spaces, Helly graphs
3. Features of Helly graphs
4. Helly groups: examples and properties
5. Further topics

Graphs

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A *tree* is a connected graph without cycles.

Group action

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Example

A *trivial* action $G \to \{1\}$.

Motto: In Geometric Group Theory we study groups via their actions on spaces equipped with some geometry.

Geometric Group Theory

...so better the actions be nice.

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Motto II: A group G and a space acted geometrically upon G look “alike”.

Cayley graph

Definition (Cayley graph)

For a group G and its generating set S, the *Cayley graph* $\mathrm{Cay}(G, S)$ has the vertex set G and edges of the form $\{g, gs\}$, for $g \in G$ and $s \in S$.

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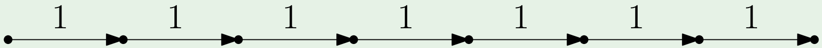
$\mathrm{Cay}(\mathbb{Z}, \{1\})$



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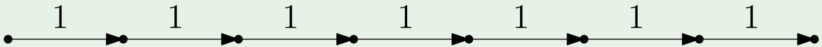
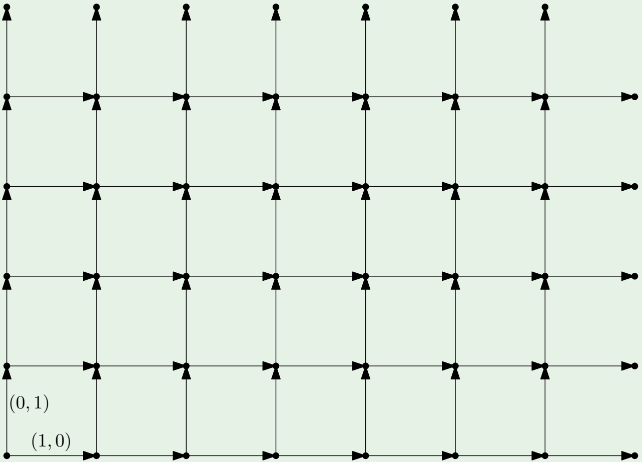


A group acts geometrically, and freely by automorphisms (preserving types of edges) on its Cayley graph via $h \cdot g = hg$.

Cayley graph

Example

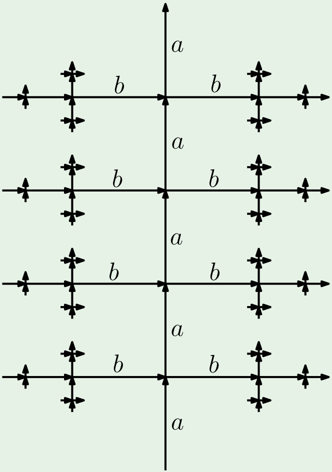
$\mathrm{Cay}(\mathbb{Z}^2, \{(0,1),(1,0)\})$



Cayley graph

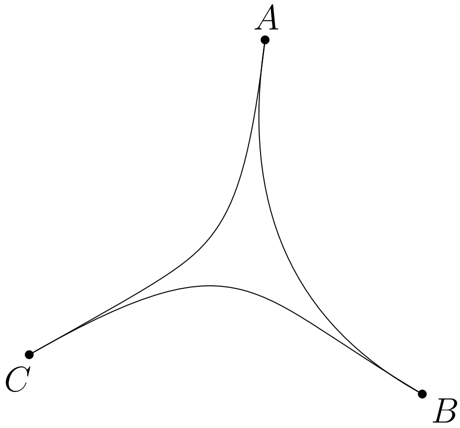
Example

Free group: $\mathrm{Cay}(F(a,b),\{a,b\})$



Gromov hyperbolicity

A geodesic triangle ABC is δ-*thin* if every its side lies in the δ-neighbourhood of the union of the other two sides.



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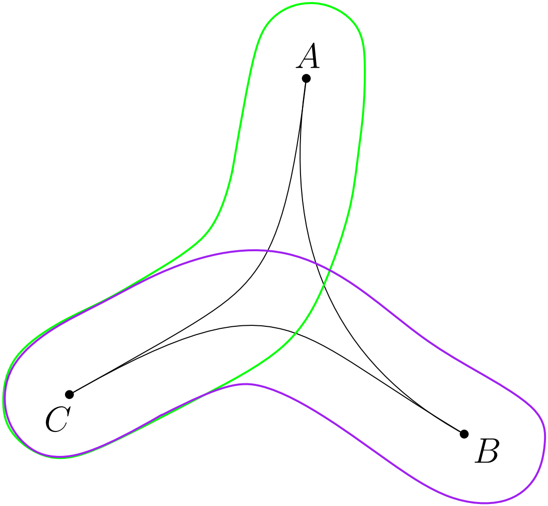

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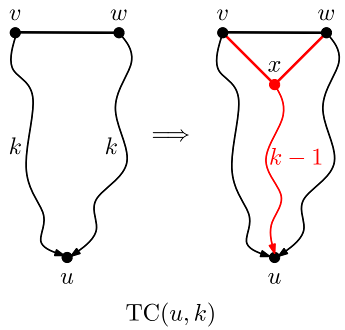
Example

$\mathbb{Z}^2$ is not hyperbolic.

Weakly modular graph

Definition (Triangle condition)

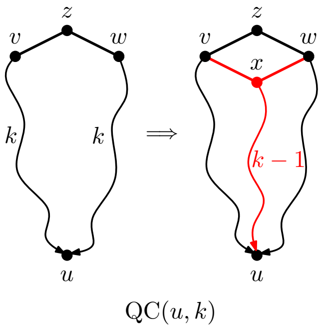
A (simplicial) graph satisfies the *triangle condition at distance* $k > 0$ *with respect to a vertex* u, denoted $\mathrm{TC}(u, k)$, if for any two vertices v, w with $1 = d(v, w) < d(u, v) = d(u, w) = k$ there exists a common neighbor x of v and w such that $d(u, x) = k - 1$.



Weakly modular graph

Definition (Quadrangle condition)

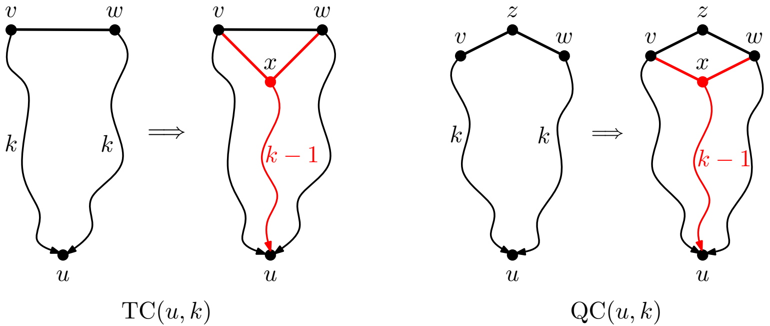
A (simplicial) graph satisfies the *quadrangle condition at distance* $k > 0$ *with respect to a vertex* u, denoted $\mathrm{QC}(u,k)$, if for any three vertices v, w, z with $d(v,z) = d(w,z) = 1$ and $2 = d(v,w) \leq d(u,v) = d(u,w) = d(u,z) - 1 = k$, there exists a common neighbor x of v and w such that $d(u,x) = k - 1$.



Weakly modular graph

Definition (Weakly modular graph)

A (simplicial) graph is *weakly modular* if it satisfies conditions TC(u, k) and QC(u, k) for all u and k.



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An *interval* $I(v, w)$ between vertices v, w in a graph Γ is the set of all vertices on geodesics from v to w (that is, u such that $d(v, w) = d(v, u) + d(u, w)$).

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A graph is *median* (aka 1-*skeleton of a CAT(0) cubical complex*) if for any vertices u, v, w the intersection $I(u, v) \cap I(v, w) \cap I(w, u)$ is a single vertex (called the *median of* u, v, w).

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Example

Tree
1-skeleton of the standard cubulation of $\mathbb{E}^n$

Helly property

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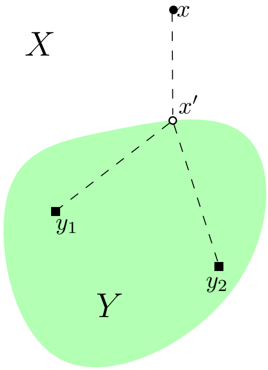
Example (Helly families)

1. axis-parallel boxes in $\mathbb{R}^n$
2. finite subtrees of a tree
3. a finite family of half-spaces of a $\mathrm{CAT}(0)$ cube complex.

Helly property - examples

Example (Gated subsets)

A subset Y of a metric space (X, d) is *gated* if for every point $x \in X$ there exists a vertex $x' \in Y$, called the *gate* of x, such that $x' \in I(x, y)$, for every $y \in Y$. A finite family of gated subsets has the Helly property.



Example (Intervals in lattices)

A *lattice* is a poset $(P, \leqslant)$ with g.l.b. (called *meet*) and l.u.b. (*join*) for each pair of elements. An *interval* in a lattice is a subset of the form $\{x | a \leqslant x \leqslant b\}$. A finite family of intervals in a lattice has the Helly property.

Injective metric spaces

Let (X, d) be a geodesic metric space

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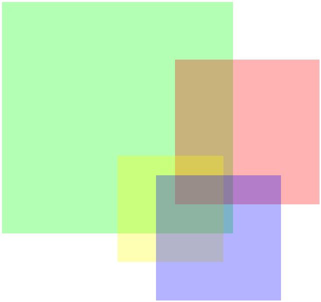
Example

1. $(\mathbb{R}^n, d_\infty)$
2. an $\mathbb{R}$-tree
3. $(\mathbb{R}^2, d_2)$ is not injective!



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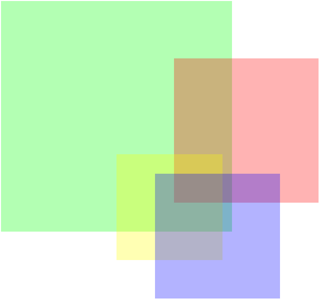


Remark

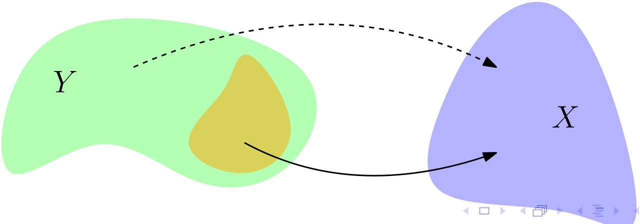
Actually, the definition of an injective space above is not so proper...

Injective spaces

Theorem (Characterizations of injectivity)

Let (X, d) be a geodesic metric space. TFAE:

1. *X is injective*
2. *X is hyperconvex*
3. *[Aronszajn-Panitchpakdi, 1956] (Y, X) has the extension property, for every metric space Y (for the category of metric spaces with 1-Lipschitz maps)*
4. *X is an absolute retract (for the category of metric spaces with 1-Lipschitz maps)*



Injective hull

Definition (Injective hull)

An *injective hull* (or *tight span*, or *injective envelope*, or *hyperconvex hull*) of (X, d) is a pair $(e, E(X))$ where $e\colon X \to E(X)$ is an isometric embedding into an injective metric space $E(X)$, and such that no injective proper subspace of $E(X)$ contains $e(X)$. Two injective hulls $e\colon X \to E(X)$ and $f\colon X \to E'(X)$ are *equivalent* if they are related by an isometry $i\colon E(X) \to E'(X)$.

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Theorem (Isbell 1964)

Every metric space (X, d) has an injective hull and all its injective hulls are equivalent.

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Remark

Injective hulls were rediscovered by Dress in 1984, Chrobak-Larmore in 1994...

Isbell's construction

Let (X, d) be a metric space. Consider the space $\mathbb{R}^X$ of real-valued functions with the supremum metric $d(f, g) = \sup_{x \in X} |f(x) - g(x)|$.

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The *Kuratowski embedding* $e \colon X \to \mathbb{R}^X \colon x \mapsto d(x, \cdot)$ is an isometric embedding.

Isbell's construction

Let (X, d) be a metric space. Consider the space $\mathbb{R}^X$ of real-valued functions with the supremum metric $d(f, g) = \sup_{x \in X} |f(x) - g(x)|$.

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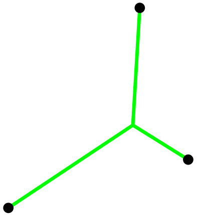
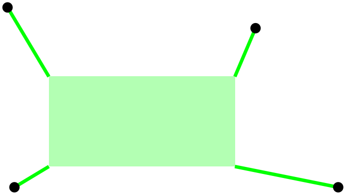
$(e, E(X))$ is the injective hull of X.

Injective hulls - examples





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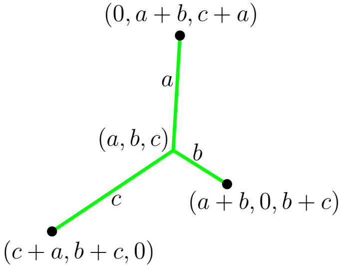

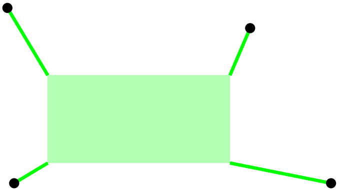

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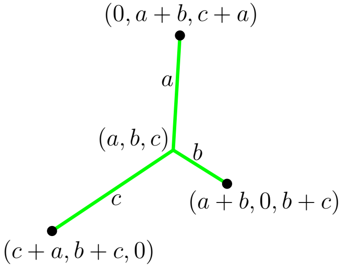

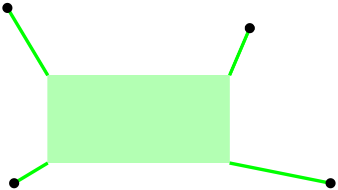

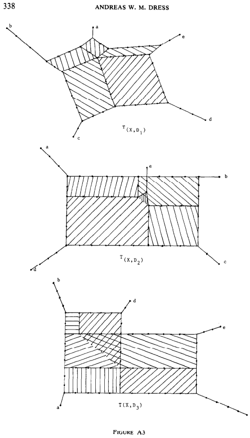
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FIGURE A3

Geodesic bicombing

Definition (Geodesic bicombing)

A *geodesic bicombing* on a metric space (X, d) is a map

$$\sigma\colon X \times X \times [0,1] \to X,$$

such that for every pair $(x, y) \in X \times X$ the function $\sigma_{xy} := \sigma(x, y, \cdot)$ is a constant speed geodesic from x to y.

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It is called *reversible* if $\sigma_{xy}(t) = \sigma_{yx}(1-t)$ for all $x, y \in X$ and $t \in [0,1]$.

End of Lecture 1