

Exercises for “Helly graphs and groups”

Masterclass “Topics in Geometric Group Theory”

Damian Osajda

List 1

- (1) Show that the triangle-square complex of a weakly modular graph, that is, the complex obtained by attaching triangles to all induced 3-cycles, and quadrangles to all induced 4-cycles is simply connected.
- (2) Show that median graphs are weakly modular.
- (3) Show that convex subsets of median graphs are gated. Show that intervals in median graphs are gated.
- (4) Show that the family \mathcal{F} of subsets of the space X has the Helly property, for:
 - (a) \mathcal{F} being the family of axis-parallel boxes in $X = \mathbb{R}^n$;
 - (b) \mathcal{F} being the family of finite subtrees of a tree X ;
 - (c) \mathcal{F} being the family of balls in $X = (\mathbb{R}^n, d_\infty)$;
 - (d) \mathcal{F} being a finite family of gated subsets of a metric space X ;
 - (e) \mathcal{F} being a finite family of intervals in a lattice X .
- (5) Show that *Helly graphs*, that is, graphs in which the family of balls satisfies the Helly property, are weakly modular.
- (6) A vertex v of a graph Γ is *dominated* if there exists a vertex w such that every neighbour of v is also a neighbour of w (in particular v is a neighbour of w).

A finite graph Γ is *dismantlable* if there exists an ordering v_0, v_1, v_2, \dots of its vertices such that for every i the vertex v_i is dominated in the subgraph of Γ induced by v_0, v_1, \dots, v_i .

Show that finite Helly graphs are dismantlable.
- (7) Show that any group action on a finite dismantlable graph has a fixed clique.
- (8) The *flag completion* of a graph Γ is a flag simplicial complex with 1-skeleton Γ .

Show that the flag completion of a finite dismantlable graph is contractible.