Exercises for "Helly graphs and groups"

Masterclass "Topics in Geometric Group Theory"
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List 0

(1) Show that an action by automorphism on a graph is proper iff stabilizers are finite.

(2) A map (not necessarily continuous) $F\colon (X, d_X) \to (Y, d_Y)$ between two metric spaces is a *quasi-isometric embedding* if there exist constants $C, D > 0$ such that

$$\frac{1}{C} d_X(x, x') - D \leqslant d_Y(F(x), F(x')) \leqslant C d_X(x, x') + D$$

for all $x, x' \in X$.

The isometric embedding $F\colon X \to Y$ is a *quasi-isometry* if there exists $E > 0$ such that for every $y \in Y$ there is $x \in X$ with $d_Y(y, F(x)) \leqslant E$.

Suppose that a group G generated by a finite set S acts geometrically on a metric space (X, d_X). Show that then (X, d_X) and (G, d_S) are quasi-isometric.

(3) Show that given two finite generating sets S, S' of a group G, the metric spaces (G, d_S) and $(G, d_{S'})$ are *bilipschitz equivalent*, that is, there exists a quasi-isometry between them with the additive constant $D = 0$.

(4) Draw all the Cayley graphs of the cyclic group C_5 of order 5. Draw a few Cayley graphs of $\mathbb{Z}$ (integers with addition) and a few Cayley graphs of F_2 (the free group of rank 2).

(5) Show that if $\mathrm{Cay}(G, S)$ is a tree then $G = F(S)$.

(6) Draw a Cayley graph Γ of $\mathbb{Z}$ and a Cayley graph Γ' of $\mathbb{Z}^2$ having the following property. Combinatorial balls of radius 7 in Γ and Γ' are isomorphic.

(7) Prove the Sabidussi Theorem: A graph Γ is a Cayley graph of a group G iff it admits a free transitive action of G by graph automorphisms.

(8) *Examples of groups.* Draw Cayley graphs of the following groups.
   (a) Baumslag-Solitar group $BS(2;1) = \langle a, b \mid ba^2b^{-1}a^{-1} \rangle$.
   (b) Heisenberg group $H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$.
   (c) The fundamental group of the surface of genus 2.
   (d) Right-angled Coxeter group $\langle a, b, c, d \mid a^2, b^2, c^2, d^2, [a, b], [c, d] \rangle$.
   (e) Right-angled Artin group $\langle a, b, c, d \mid [a, b], [c, d] \rangle$.

(9) Show that the 1-skeleton of a triangulation of a plane such that every vertex is contained in exactly 7 triangles is a hyperbolic graph.

(10) Show that any action of a finite group on a tree fixes a point.

(11) Show that any action of a finitely generated torsion group (that is, every element has finite order) on a tree fixes a point.