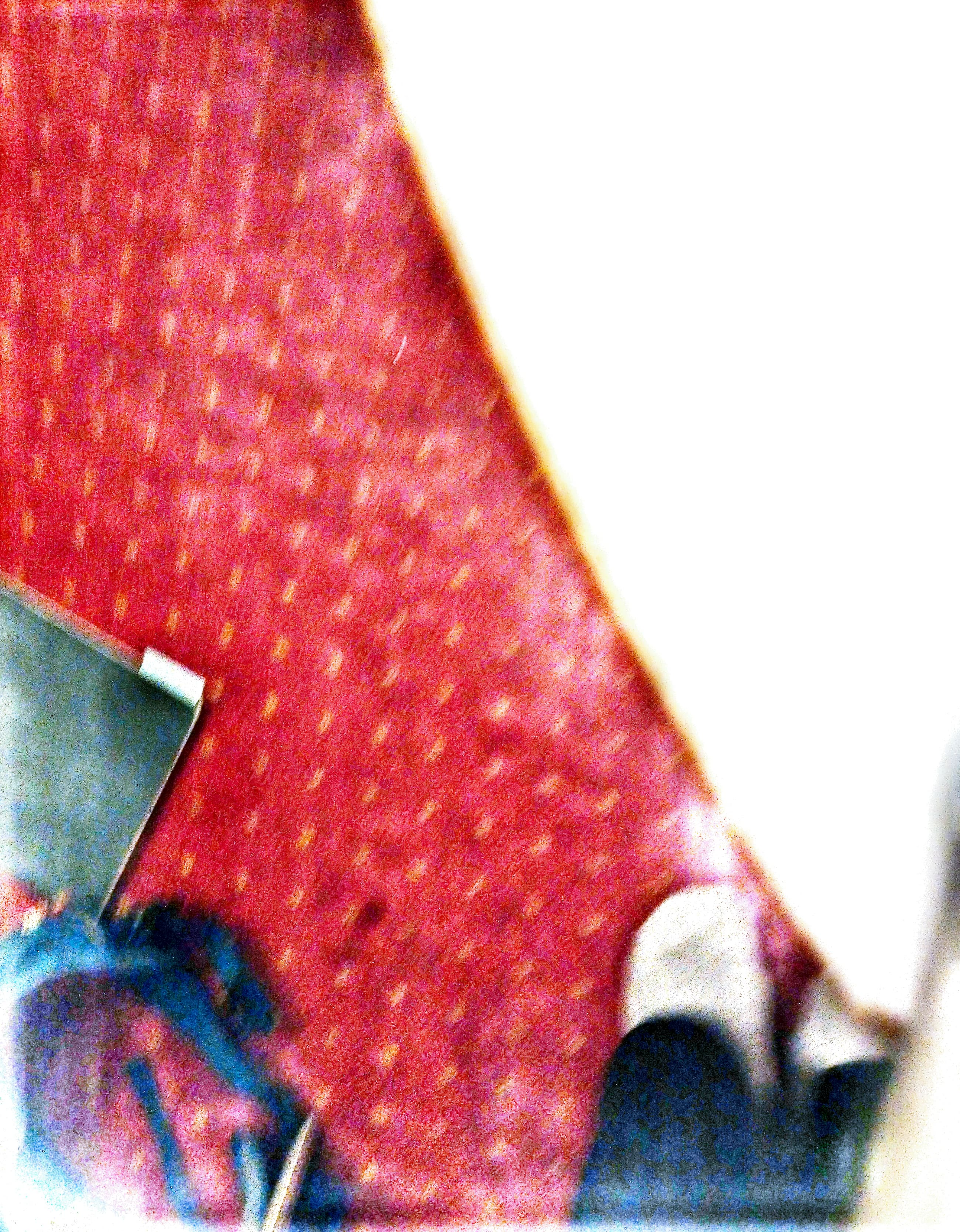


Part 4 Connection with CAT(0) geometry

① CAT(0) spaces

Given (X,d) a geodesic metric space. Let x,y,z be a triple of pts in X and $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ three connecting geodesic segments forming a triangle $\Delta(x,y,z)$

Def. A Euclidean (spherical) comparison triangle for $\Delta(x,y,z)$ is a triple $(\bar{x},\bar{y},\bar{z})$ of points in $(\mathbb{R}^2, d_{eucl})$ (resp. on (S^2, d)) s.th.

$$d(x,y) = d_{eucl}(\bar{x},\bar{y}) \quad \text{and} \quad d(y,z) = d(\bar{y},\bar{z}).$$
$$d(x,z) = d_{eucl}(\bar{x},\bar{z})$$

Exercise: prove that every triangle has a Euclidean comparison triangle and if the circumference is $< 2\pi$ also a spherical one.

Def The space (X,d) is <u>CAT(0)</u> if — CAT(1) with spherical comp triangles

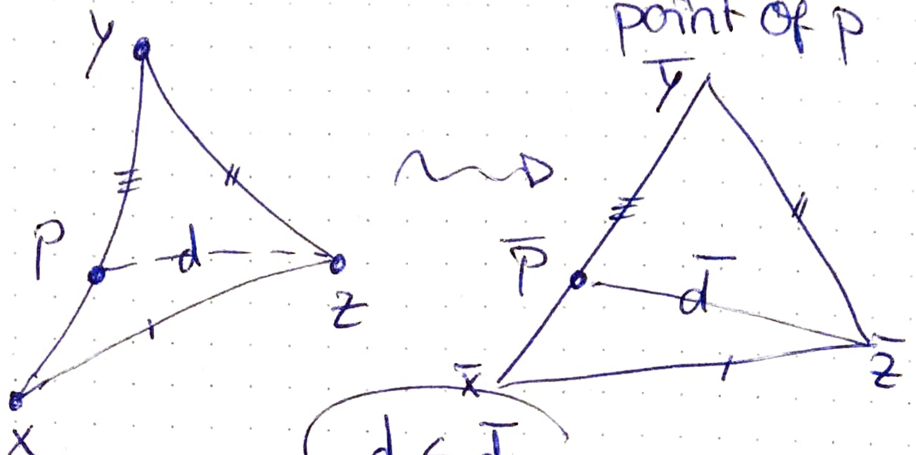
for every triangle in X on pts x,y,z

and every point p on $\gamma_{x,y}$

one has $d(z,p) \leq d_{eucl}(\bar{z},\bar{p})$

↑ comparison point of p

<u>slogan:</u> triangles in a CAT(0) space are thinner / at most as thick than their comparison triangles in $\mathbb{R}^2$



$d \leq \bar{d}$

$\bar{p}$ s.th. $d(x,p) = d_{eucl}(\bar{p},\bar{x})$

In general CAT(0) condition is hard to test ⇝ not so much for cube cplxes.

Def. We say (X,d) is <u>locally CAT(0)</u> iff

$\forall x \in X \; \exists r \in \mathbb{R}_{>0}$ s.th. $B_r(x)$ is CAT(0)

<u>Examples:</u>

- $\mathbb{R}^n, d_{eucl}$
- complete Riem. mfld M is CAT(0) $\iff$ M has non-pos. sectional curvature
- trees, (affine) buildings

Elementary properties

(i) CAT(0) spaces have unique geodesics and are contractible

(ii) every local geodesic is a (global) geodesic

(iii) convex subsets of CAT(0) spaces are again CAT(0).
Same for products, cauchy completion asymptotic cones, ultraproducts, or gluings along convex subsets.

Analog of the classic Cartan-Hadamard thm holds:

Thm (Ballmann, Alexander-Bishop)

(X,d) complete, connected and locally CAT(0), then

$\exists!\ \tilde{d}$ on $\tilde{X}$ = univ cover s.th. $\tilde{X} \to X$ is a local isometry and

$(\tilde{X}, \tilde{d})$ is a CAT(0) space.

Thm Bruhat-Tits fixed pt Thm

(X,d) complete, CAT(0) space, $G < \mathrm{Isom}(X)$.
If G setwise stabilizes a bounded set in X, then

$$X^G \neq \emptyset \text{ and convex.}$$

$$= \{x \in X \mid gx = x \ \forall g \in G\} = \text{fixed pts of } G \text{ on } X$$

is a consequence of the fact that bdd sets in CAT(0) spaces have unique centers
i.e. pts $p \in Y$ s.th. $B_r(p) \supset Y$
where $r = \inf_{x \in Y} (\sup_{a \in Y} d(x,a))$
$= \inf\{r \in \mathbb{R}^+ \mid Y \subset \overline{B}_r(x) \text{ for some } x \in X\}$

<u>Ex/Rmk</u> nondiscrete Buildings arise naturally as asymptotic cones of symm spaces of noncompact type

<u>Thm</u> (Kleiner 1999) — "local str. of length spaces with curv. bdd from above"

X locally compact CAT(0) space of geometric dimension n. If any pair of pts is in a common <u>n-flat</u>, i.e. isometrically embedded copy of $\mathbb{R}^n$, then X is the metric realization of an affine building.

Here <u>geom. dim</u> = sup of the topol. dim of all compact subsets $K \subset X$.

② How to test for CAT(0)?

<u>Gromov's link condition</u>

An M_κ-polyh. cplx with fin. many shapes of cells is CAT(0) if and only if $\forall$ vertices v in the cplx X $Lk_X(v)$ is CAT(1)

X a cube cplx: CAT(0) $\Leftrightarrow$ $Lk_X(v)$ flag

<u>Bowditch's criterion</u>

X locally CAT(1) then
X CAT(1) $\Leftrightarrow$ every loop of length $< 2\pi$ in X is shrinkable

I used this to prove B_6 is a CAT(0) grp.

③ CAT(0) groups

Def. A grp G is a <u>CAT(0) grp</u> if it admits an action on a CAT(0) space that is proper, cocompact and by isom.

$\forall x \in X \; \exists \; B_r(x)$ s.th. $|\{ B \cap gB \neq \emptyset,\ g \in G\}| < \infty$

geometric action

Examples:

Coxeter grps, RAAGs, small cancell. grps, braid grps for $n \leq 7$,

<u>open</u>: Artin grps (Conj. by Ruth Charney)

<u>Non-example</u>: $Aut(F_3)$ (Gersten 1994) autom. grp of a free grp, $Out(F_n)$ $n \geq 3$ | MCG of genus $g \geq 3$ not CAT(0)

Properties:

Thm A CAT(0) group G satisfy:

(i) They have only fin. many conj. classes of finite subgroups.

(ii) G is finitely presented

(iii) Every solvable subgrp is virtually $\mathbb{Z}^n$

(iv) word- and conjugacy problem are solvable

<u>open</u>: isomorphism problem

(v) $G \curvearrowright X =$ CAT(0) geom., then G is hyperbolic if and only if X does not contain a 2-flat

i.e. $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ geom., $\mathbb{Z}^2$ not hyperb. is the only obstruction

Bieberbach Thm

G f.g. then

G acts geom. on $\mathbb{E}^n$ $\Longleftrightarrow$ G contains a f.i. subgrp $A \cong \mathbb{Z}^n$

The Flat Torus Thm

$G \underset{\text{geom}}{\curvearrowright} X$ CAT(0), $\exists$ subgrp $A \cong \mathbb{Z}^n$
then $\exists$ isom. emb. of $\mathbb{E}^n \overset{f}{\hookrightarrow} X$
s.th. $f(\mathbb{E}^n)$ is stabilized (setw.) by A.
(A has a torus action on $f(\mathbb{E}^n)$).

Important open conjecture:

see Stefan Stadler's papers for most recent results

Ballmann's rank rigidity conjecture

X locally comp., geod. complete CAT(0)
Suppose for a grp $G < \mathrm{Isom}(X)$ that has
(*) $\underbrace{\Lambda(G)}_{}= \partial_\infty X$:= ideal bdry
limit set, i.e. accumulation pts of a G-orbit on X
then if $\mathrm{diam}(\partial_T(X)) = \pi$, then X is a Eucl. building of rk 2, nonc. symm space or a non-trivial product

Rmk: (*) is satisfied by geometric actions