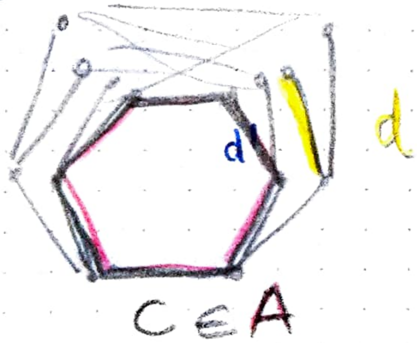
Part 3 Retractions and BN-pairs

Every building Δ (simpl. cpl. + (B0)-
admits retractions: pick $c \in A$ chamber.
Define $\rho_{A,c} : \Delta \longrightarrow A$ to be the
map that assigns to a chamber d in Δ the image $\iota(d) \in A$ under the isomorphism $\iota : A' \longrightarrow A$ existing by (B2), where $A' \ni c, d$.

Ex:

a)



$\rho_{A,c}(d) = d'$

b)

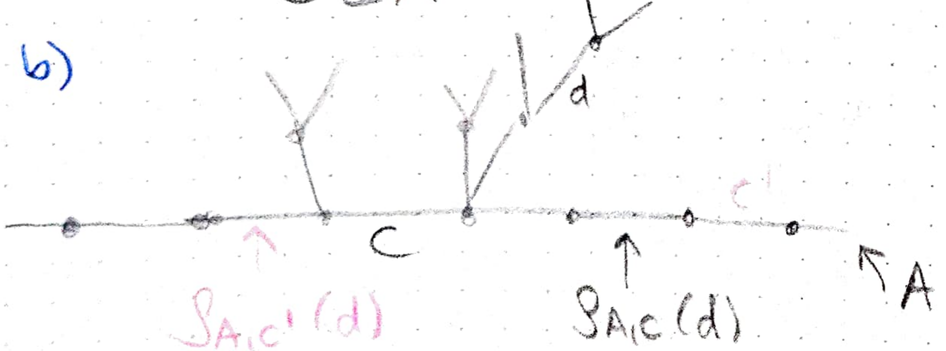
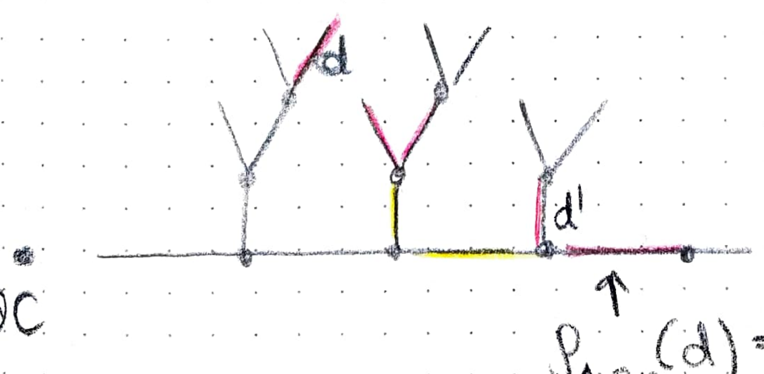


Image / retraction depends on both c and A

Affine buildings allow for a second type of retraction: (In fact $n+1$ types, $n=\mathrm{rk}$)
Fix an apartment A in Δ and a chamber ∂C at ∞ of A.

<u>Fact</u> $\forall$ chambers d in Δ there exists an apartment A' containing d and s.th. $\partial A'$ contains ∂C.

This allows us to define <u>retractions from ∞</u>:
Define $\rho_{A,\partial C}(d) := \tilde{c} \in A$
where $\tilde{c} \in A$ is the isometric image of d under $\ell : A' \to A$ (existing by (B2)).



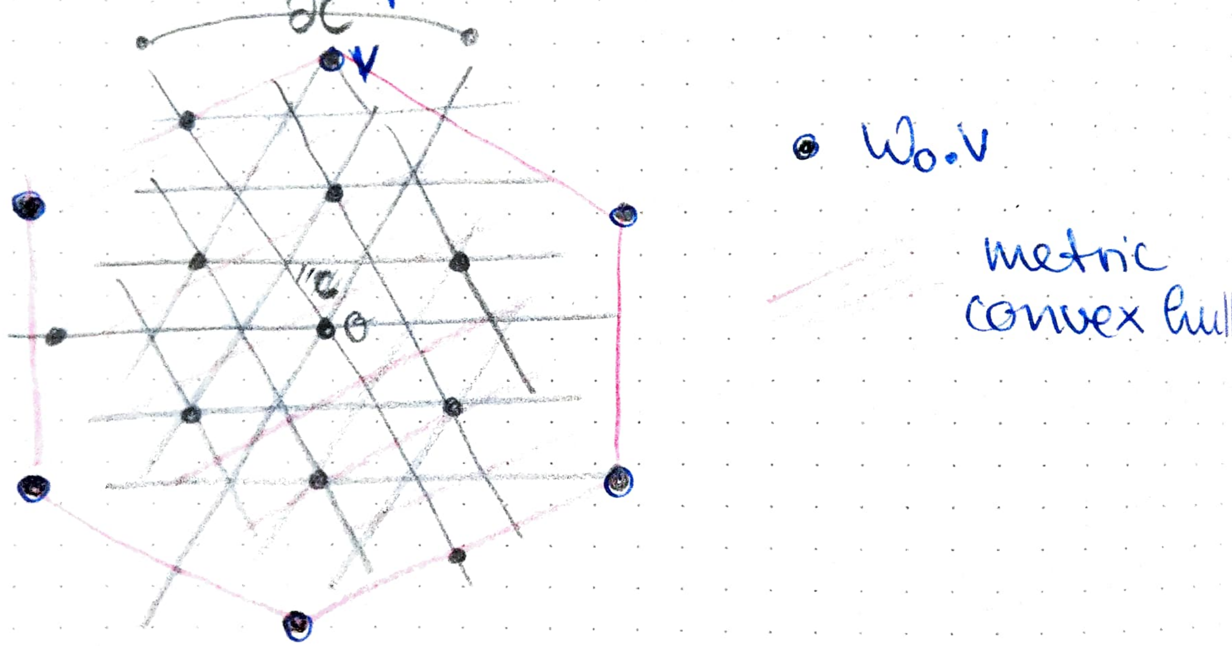
<u>How do these retractions interact?</u>

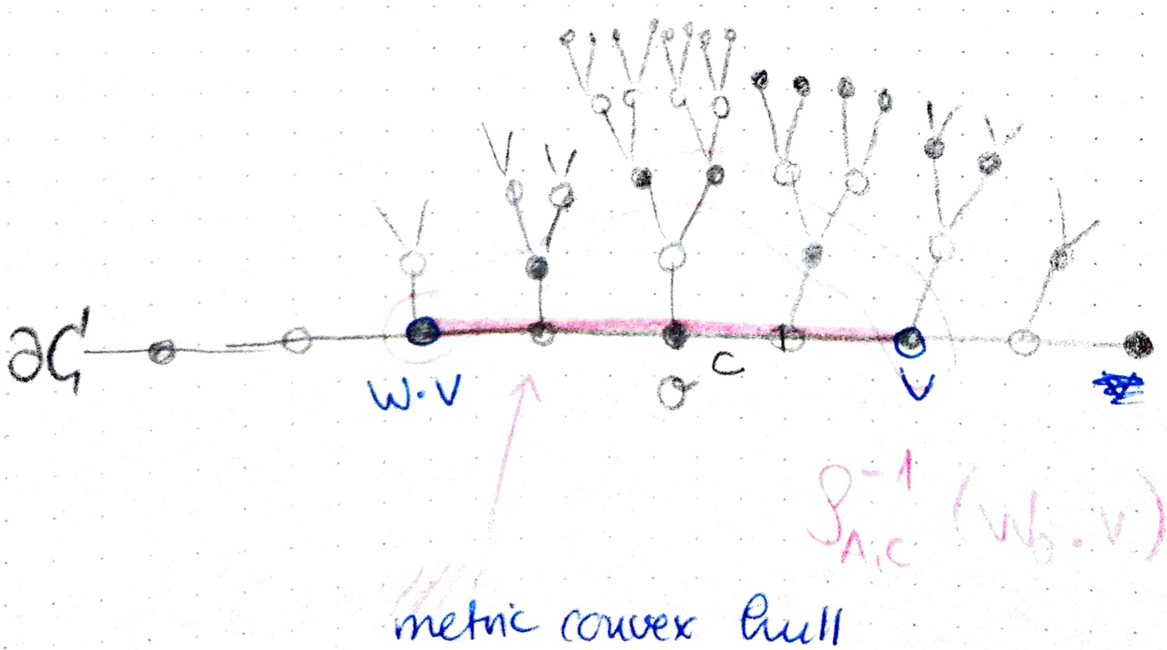
<u>Thm</u> (Geometric convexity thm, Schwer 2009)
Δ affine bldg, A apmt in Δ, $C \in A$, $\partial C \in \partial A$
then $\rho_{A,\partial C}(\rho_{A,C}^{-1}(W_0 . v)) = \mathrm{conv}^*(W_0 . v)$
vertex in A

here $\mathrm{conv}^*(W_0 . v)$ is the set of all vertices in the metric convex hull of $W_0 . v$ having the same color as v.

$W_0 = \mathrm{Stab}_W(o)$ = spherical Weyl grp

and o is a special vertex of C.





Idea for proof:

Consider a minimal gallery $\gamma_v : o \rightsquigarrow v$
- look at the orbit $W_0 \cdot \gamma \ni \tau$
- look at pre-images $\rho_{A,C}^{-1}(\tau)$
- track what $\rho_{A,\partial C}$ does to the preimages, which turn out to be minimal galleries from o to elements in $\rho_{A,C}^{-1}(W_0 \cdot v)$ in Δ.
- this naturally produces positively folded galleries wrt $\phi_{\partial C}$

$\rightsquigarrow$ remains to prove that

$$Sh_{\phi_{\partial C}}(v) := \left\{ \begin{array}{l} \text{endpts of } \phi_{\partial C} \\ \text{pos. folded galleries} \\ \text{of type } \gamma_v \end{array} \right\} = \operatorname{conv}^*(W_0 \cdot v)$$

„ □ "

Def. Given an orientation ϕ and a simplex σ in Σ define

$$Sh_\phi(\sigma) = \left\{ \begin{array}{l} \text{end-simplices of type } \sigma \\ \text{of galleries of type } \gamma_\sigma \\ \phi \text{ pos. folded} \end{array} \right\}$$

the ϕ-shadow of σ.

Typically σ would be a vertex or a chamber.
Type γ_σ means: type of a minimal gallery connecting o (resp. C_0) with σ in the cox. cplx Σ.

<u>Properties</u>: • Shadows are recursively computable
• these shadows encode certain double coset intersections in reductive grps

In order to see this we need to link retractions to grps, in particular to BN-pairs:

<u>Definition</u> A <u>BN-Pair</u> for a grp G is a pair B, N of subgroups of G s.th.
(BN0) G is generated by $B \cup N$
(BN1) $T := B \cap N$ is normal in N and the group $W := N/T$ has a generating set $S = \{s_1, \ldots, s_n\}$ s.th.
(BN2) $\forall w \in W$ and $\forall s_i \in S$
$$BwB \cdot Bs_iB \subseteq BwB \cup Bws_iB$$

we have a <u>strong</u> BN-pair if in addition
(BN3) $\forall s_i \in S: \quad s_i B s_i^{-1} \neq B$.

<u>Rmk</u>
- (BN3) corresp. to the bldg being thick
- S is uniquely determined by (BN0-3)
- One can prove that W is a Coxeter group (see eg Brown V.2.A)

Buildings give rise to BN-pairs.
BN-pairs allow to construct buildings.

Example: Fano plane (B,N), SL_2 tree (I,N)

<u>Thm</u> A (Tits)

Let Δ be a building of type (W,S) with apartment system $\mathcal{A}$.
Let $G \curvearrowright \Delta$ color preserving and strongly transitive wrt $\mathcal{A}$.
Fix $A_0 \in \mathcal{A}$ and a chamber $C_0 \in A$.
Put: $\quad B :=$ pointw. stab. in G of C_0
$\quad\quad\;\; N :=$ setw. stab. in G of A_0
Then (B,N) is a BN-pair, i.e. satisfies (BN0) to (BN2). In addition (BN3) holds if the building Δ is <u>thick</u>, i.e. every panel is contained in at least 3 chambers.

Thm B (Tits)

Suppose G is a grp with a BN-pair.
Put $W := N/T$ with $T := B \cap N$.
Then there exists a building $\Delta = \Delta(B,N)$ of type (W,S) s.th.

(i) the set of chambers is given by $\{gB \mid g \in G\}$

(ii) two chambers are i-adjacent
$gB \sim_i hB \iff g^{-1}h \in P_i$
where $P_i = B \sqcup Bs_iB$

(iii) there is a W-valued distance function on Δ and
$\delta(gB, hB) = w \iff g^{-1}h \in BwB$

If (BN3) holds Δ is thick.
Put $C_0 := B$ and $A_0 := \{wC_0 \mid w \in W\}$
then G acts strongly transitive on Δ wrt $\mathcal{A} := \{gA_0 \mid g \in G\}$.
The subgrp N stabilizes A_0 setwise.

Rmk Buildings obtained in this way are called of algebraic origin.

Groups with BN-pair satisfy the
<u>Bruhat decomposition:</u>

<u>Lemma:</u> If G has a BN-pair, then

$$G = \bigsqcup_{w \in W} BwB$$

<u>resp.</u>

$$G = \bigsqcup_{w \in W} IwI \quad \text{in the affine case}$$

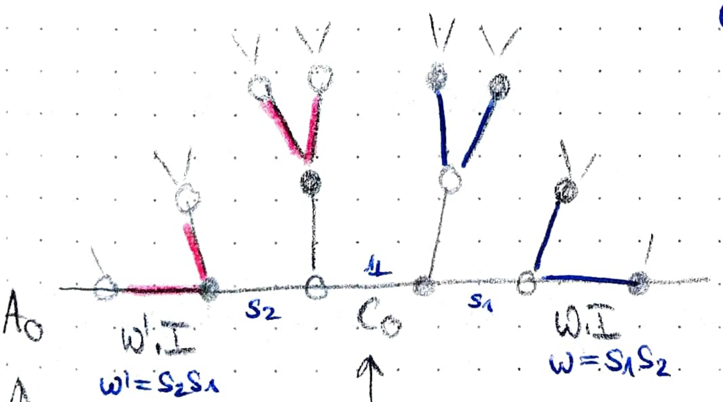
one chamber

permutes apartments to cover all of Δ — one apartment

<u>Geometric interpretation of Bruhat decomp:</u>

1) $SL_2(K)$, $v: K \to \mathbb{Z}$, k = res. field e.g. $K = \mathbb{Q}_p$

$C_0 \mathrel{\hat{=}} I$ = Iwahori subgrp as in lecture 2, "affine B"



the double coset $IwI \mathrel{\hat{=}} w.C_0$ (and more generally BwB) is the collection of all chambers in the I-orbit of $w \cdot C_0$

Bruhat decomp. says, that every
$d = g.C_0$ is contained in a unique
such orbit!

— = I orbit of $wI = s_1 s_2 I$

— = -||- $s_2 s_1 I$

Now the retraction $\rho_{A_0,C_0} : \Delta \to A_0$ has the property that $\rho_{A_0,C_0}^{-1}(wI) = I$

One could define (equiv. to geom. def. earlie[r]) the image of C under ρ_{A_0,C_0} is the ~~[illegible]~~ unique chamber wI, $w \in W$ s.th. C is contained in the cell IwI. (resp. BwB when looking at spherical buildings).

The geometric Convexity theorem has the following consequence:

Thm (Kostant convexity for Buildings)
Schwer 2009

G a grp with an affine BN pair. Then

ppes aug. xes → $$B t K_\circ \cap K_\circ t' K_\circ \neq \emptyset$$

$$\iff t \in \mathrm{conv}^*(W_0 \cdot \underbrace{t' K_\circ}_{\text{a vertex in } A_0})$$

where $K_\circ = \mathrm{Stab}_G(0)$

and t, t' are translations in the affine Coxeter group W of G.

Summary of BT - case:

Let G be a red. grp over a field K with discrete valuation
$\nu: K \to (\mathbb{Z},+)$

We then have:
$$K \xleftarrow{\iota} \mathcal{O} \xrightarrow{pr} k$$
valuation ring, residue field

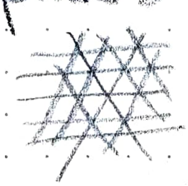
This results in 3 BN pairs:

$\underbrace{B(K), N(K)}$	$\underbrace{I, N(K)}$	$\underbrace{B(k), N(k)}$
↓	↓	↓
W_1 finite and Δ_1 spherical	W_2 infinite Δ_2 affine	W_3 finite $(=W_1)$ Δ_2 Spherical
$W_1 \cong Sym(n)$	$W_2 \cong Sym(n) \ltimes \mathbb{Z}^n$	$W_3 \cong W_1$

n = 3, p = 2:

Δ_1 = spher bldg, locally infinite, apmts $\cong$ 

Δ_2 = affine bldg, apmts $\cong$



Δ_3 = incidence graph of Fano Plane, apmts $\cong$ 

$$\Delta_1 = \partial\Delta_2 \xleftarrow[\text{at } \infty]{} \Delta_2 \xrightarrow[\text{locally at links}]{} \Delta_3 = \mathrm{lk}_{\Delta_2}([\langle e_1, e_2, e_3\rangle_{\mathcal{O}}])$$

Example

$G = SL_n(\mathbb{Q}_p)$

$\nu = \nu_p$ p-adic valuation

$K = \mathbb{Q}_p$

$\mathcal{O} = \mathbb{Z}_p$

$k = \mathbb{F}_p$

$B(*) = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$

$N(*)$ = mon mxes/$*$

$I = pr^{-1}(B(k)) = \begin{pmatrix} \mathcal{O} & & \mathcal{O} \\ & \ddots & \\ \pi\mathcal{O} & & \mathcal{O} \end{pmatrix}$