Part 2 Coxeter groups and some gallery combinatorics

① **Def.** A Coxeter group W is a group given by a presentation as follows:

$$W = \langle \underbrace{s_1, \dots, s_n}_{S} \mid s_i^2 = 1\!\!1 = (s_i s_j)^{m_{ij}} \rangle$$

where $m_{ij} \in \{2, 3, 4, \dots, \infty\}$ $m_{ii} = 1$

$m_{ij} = \infty$ means no relation

The Coxeter system (W, S) is encoded by the Coxeter matrix $M := (m_{ij})_{ij}$

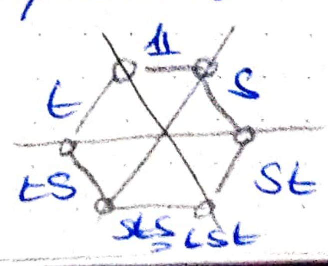
One can encode M in a graph with vertex set S and edges labeled m_{ij}.

Open problem: Which Coxeter matrices encode isomorphic groups?

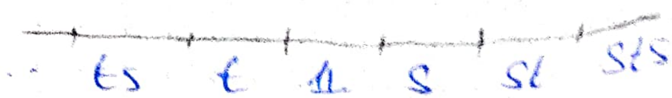
⇝ See Schwer - Santos Rego: The Coxeter galaxy

Examples:

a) $Sym(3) = \langle s, t \mid s^2 = t^2 = (st)^3 \rangle$



b) $D_\infty = \langle s, t \mid s^2 = t^2 \rangle$



Properties

a) Coxeter groups come in 3 classes:
$\underbrace{\text{spherical, affine}}_{\text{classified}}$, infinite but not affine

b) affine Coxeter grps are semidirect products of ~~of~~ a spherical Cox. grp and a translation subgrp isom to $\mathbb{Z}^n$ where $n = |S| - 1$

c) all Coxeter groups are linear, i.e. admit a faithful representation $\rho: W \to GL_n(V)$ where $V = \mathbb{R}$-VS of dimension $n = |S|$.

d) All Coxeter grps have natural simplicial complexes associated to them (Coxeter complex).
The "dual" or adjacency graph of those complexes is the Cayley graph of W wrt S.

e) Coxeter grps are CAT(0), admit a proper (but not always cocompact) action on a cubical complex and most of them are systolic.

② Construction of the Coxeter complex

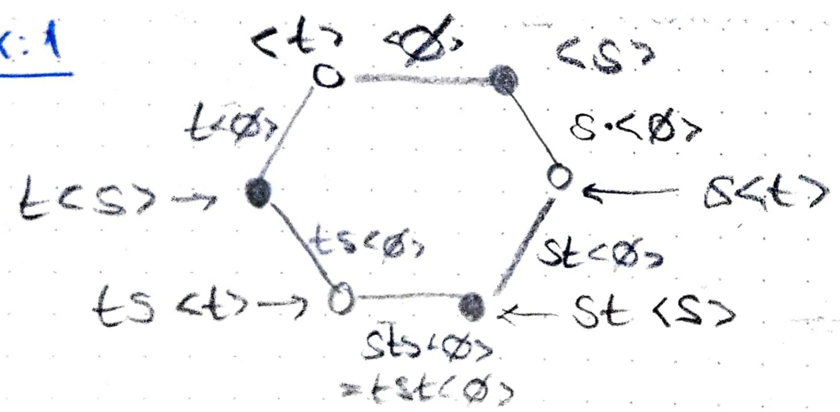
Given (W,S) Coxeter system. Consider

- special subgroups $W_{J'} := \langle S' \rangle < W$ where J' is the index-set of $S' \subset S$
- special cosets $w\langle S' \rangle$, $w \in W$, $S' \subset S$

The <u>Coxeter complex</u> $\Sigma = \Sigma(W,S)$ is the poset of special cosets ordered by reverse inclusion:

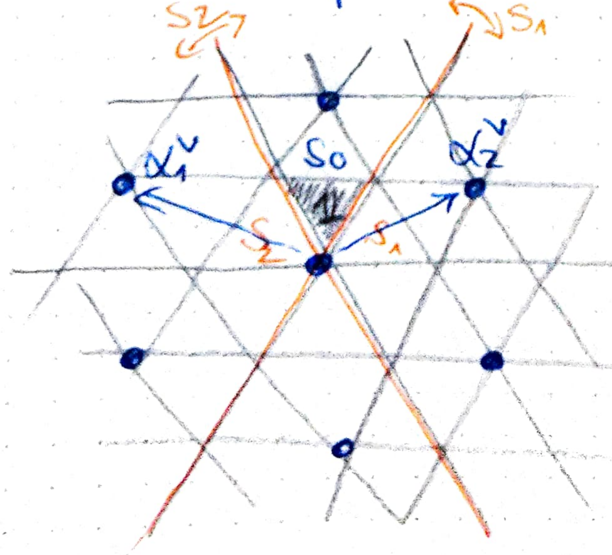
$B \leq A$ in Σ iff $B \supset A$ as subsets of W

<u>Ex: 1</u>

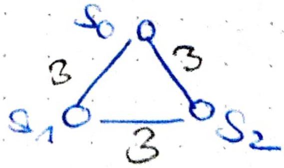


$W_0 = \langle s_1, s_2 \mid s_i^2, (s_1 s_2)^3 \rangle$

<u>Ex 2</u> The Coxeter group of type $\tilde{A}_2$ has Cox. cplx:



$W = \langle s_0, s_1, s_2 \mid s_i^2 = 1 = (s_i s_j)^3 \rangle$
$i \neq j$



- translation lattice in $\Sigma \cong \mathbb{R}^2$

$W \cong W_0 \ltimes \mathbb{Z}^2$

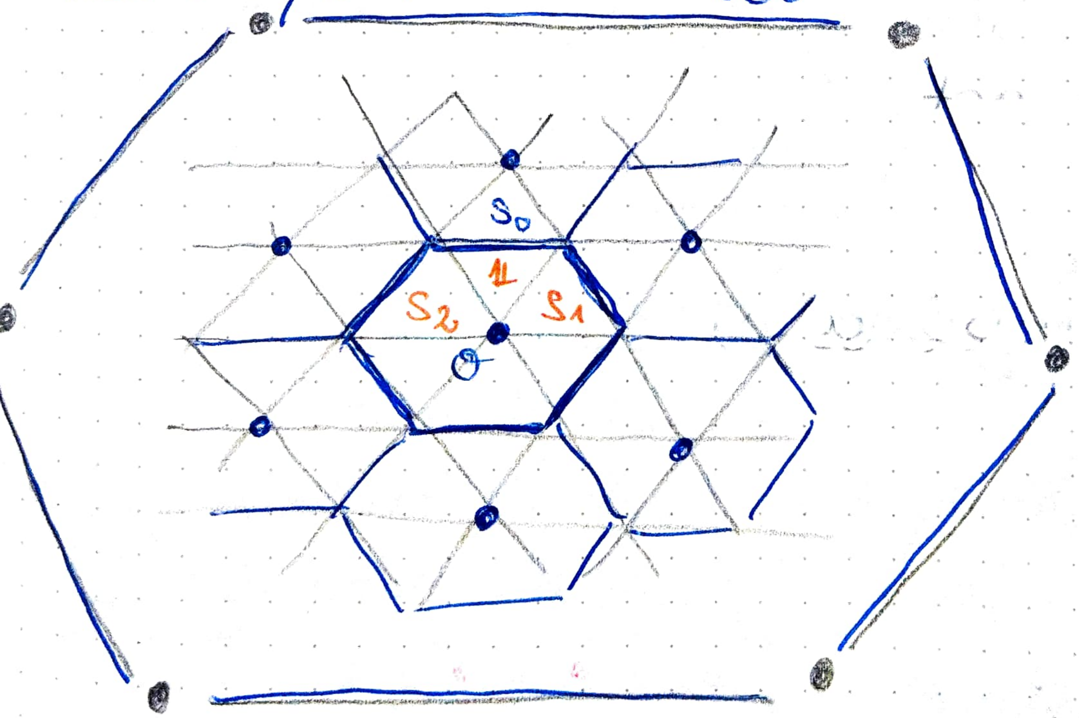
W_0: gener. by s_1, s_2

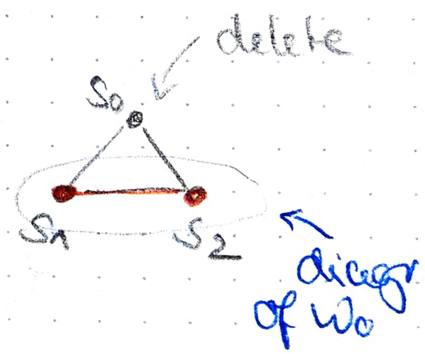
$\mathbb{Z}^2$: torus in $SL_3(\mathbb{Q}_p)$

<u>Fact/Def/Char</u>:
A Coxeter group is <u>spherical</u> / <u>affine</u> if the geom. realiz. of Σ is isom. to a sphere S^n with $n = |S|$, resp. a eucl. space $\mathbb{R}^n$, $n = |S|-1$.

<u>Fact</u>: Every affine Coxeter group W is the semidirect product of a spherical Cox. grp. W_0 of ass. type and a translation subgrp isom. to $\mathbb{Z}^n$, $n = |S|-1$.

Two ways to "see" W_0:





W_0 has a natural action on the boundary* of $\Sigma \cong \mathbb{R}^n$

* visual boundary or Gromov bdry

The link of the origin is the Coxeter cplx of W_0 and so is the simplicial structure induced on $\partial\Sigma$

$$\boxed{W = W_0 \ltimes \mathbb{Z}^n}$$

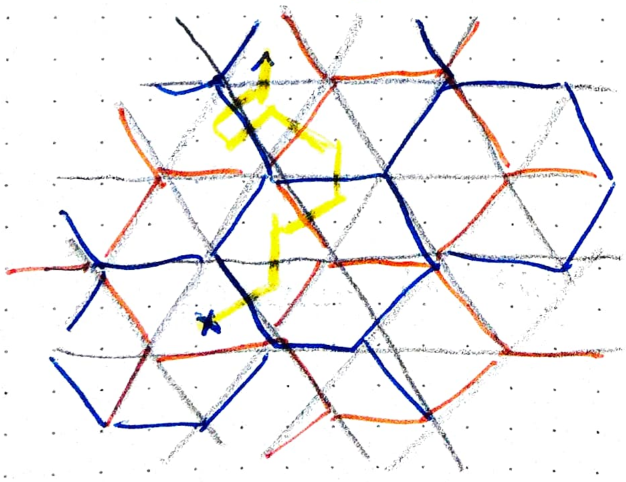
Properties: ①

a) A Coxeter complex is a thin chamber complex of rank $n = |S|$ on which W naturally acts.

b) Maximal simplices are called chambers. They are in 1:1 correspondence with elements of W.

c) faces/panels of chambers can be colored by (subsets of) the generators in S. This coloring is preserved by the W-action.

③ Walking around in Σ



A list of colors + a start chamber yields a path through Σ

This is captured in the definition of a gallery.

Def. A <u>gallery</u> in $\Sigma = \Sigma(W, S)$ is a sequence of max. simplices and codim 1 faces as follows:

$\gamma = (C_0 \supset P_1 \subset C_1 \supset P_2 \subset C_2 \supset \dots P_n \subset C_n)$

P_i panels, C_i chambers.

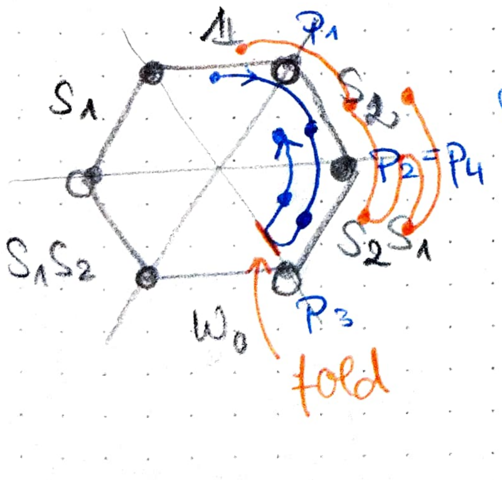
Possible: C_0, C_n any simplex, e.g. vertex.

A gallery is <u>folded</u> at i if $C_{i-1} = C_i$.

Galleries give rise to words in the generators S by listing the colors of the P_i's:

$\vec{w} = s_{i_1} \cdots s_{i_n}$ where $s_{i_j} = \text{type}(P_j)$

In order to record a fold we may put a hat on one of the letters.



gallery: ~~$\mathbb{1}, P_1, s_2, P_2, s_2 s_1, P_4, s_2$~~

$(\mathbb{1}, P_1, s_2, P_2, s_2 s_1, P_3, s_2 s_1, P_4, s_2)$

has word $s_2\, s_1\, \hat{s}_2\, s_1$

associated to ist. $\overset{\text{type}(P_1)}{\|}$ $\overset{\text{type}(P_2)}{\|}$

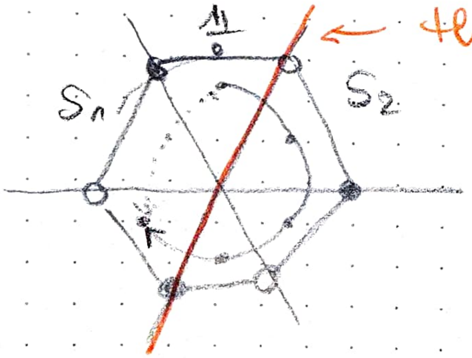
indicates the fold at P_3

<u>Fact</u>: there is a bijection between all (folded) galleries starting at a fixed chamber C_0 and all decorated words in S.

Coxeter groups can be characterized using one of three conditions
(D) deletion condition
(E) exchange condition and
(F) folding condition.

The deletion condition (D) says:
if $w = s_1 \dots s_r$ is not reduced, then there exists $i<j$ s.th. $w = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_r$

Ex: In Sym(3) the word $s_2 s_1 s_2 s_1$ is not reduced as $s_2 s_1 s_2 s_1 = \cancel{s_2}\cancel{s_2} s_1 s_2 = s_1 s_2$
⇝ can put $i=1$, $j=4$ $\quad s_1 s_2 = \hat{s}_2 s_1 s_2 \hat{s}_1$



← this wall is crossed twice
⇝ remove both crossings

The exchange condition (E) says:
given $w \in W$, $s \in S$ s.th. $l(sw) < l(w)$.
Suppose $w = s_1 \dots s_r$ is reduced
then $sw = s_1 \dots \hat{s}_k \dots s_r$ for some k.

The <u>folding condition</u> (F) says:

$w \in W$, $s, t \in S$ s.th. $\ell_S(sw) = \ell_S(w) + 1$
and $\ell_S(wt) = \ell_S(w) + 1$, then
either $\ell_S(swt) = \ell_S(w) + 2$ or $swt = w$.

<u>Rmk</u>: One can prove that for pre-Coxeter systems these conditions are equivalent and characterize Coxeter systems.

⤳ See Brown's book for details.

4) Orientations on Σ:

Every Coxeter complex contains <u>hyperplanes</u> i.e. sub-complexes that are fixed by a reflection $r \in W$.

The set of <u>reflections</u> is given by

$$R := \underbrace{\{ wsw^{-1} \mid s \in S, w \in W \}}_{\text{all conjugates of } S}$$

Complements of hyperplanes are (open) <u>half-spaces</u>.

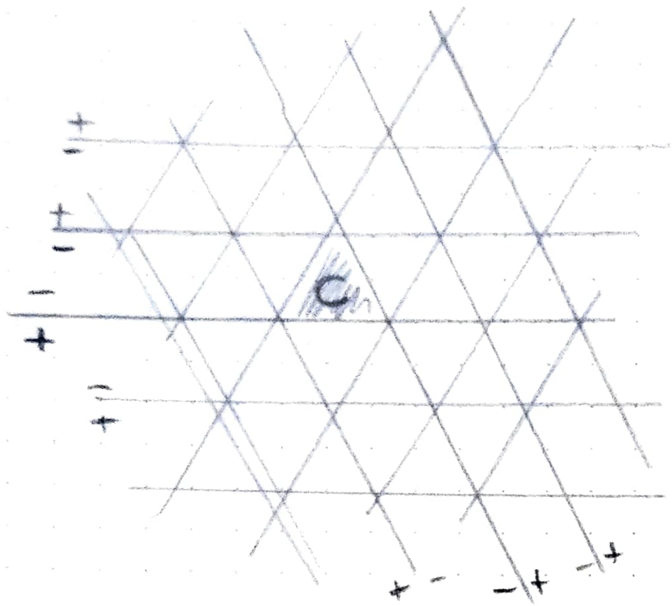
Def. An <u>orientation on Σ</u> is
a map that assigns to every half space an element of $\{\pm 1\}$.

↑ meaningless as is.

<u>Trivial</u> <u>pos</u> (neg) <u>orientation</u> is constant and equal to $+1$ (-1).

Examples

a) Orientation induced by a chamber $c \in \Sigma$:

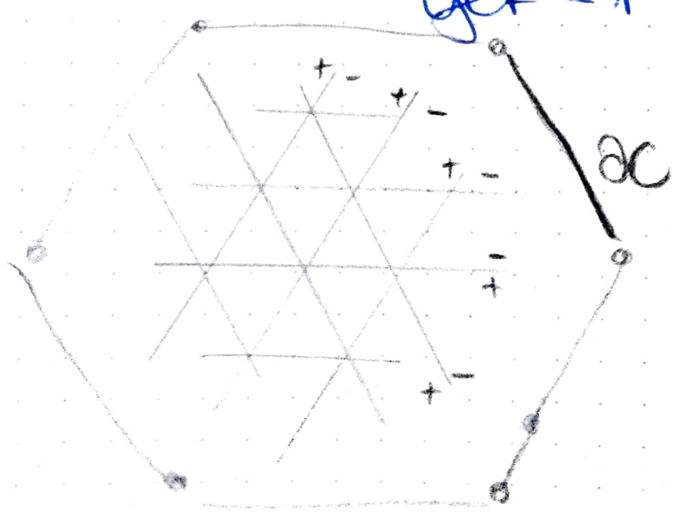


half-spaces containing c are assigned -1

otherwise $+1$

$\rightsquigarrow$ orientation ϕ_c

b) Orientation induced by a chamber ∂c at ∞: half spaces containing ∂c get -1, otherwise $+1$



periodic, i.e. all halfspaces that are nested get the same orientation

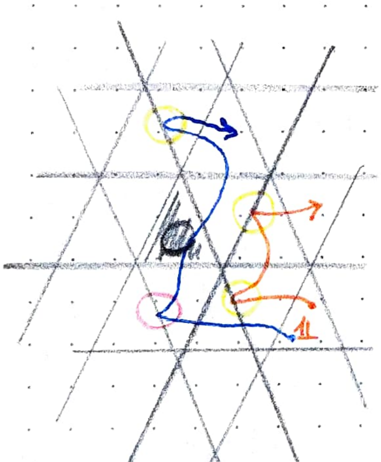
$\rightsquigarrow$ orientation $\phi_{\partial c}$

Orientations allow to refine the notion of a folded gallery:

Def. Given a gallery $\gamma = (c_0, p_1, c_1, \dots, p_n, c_n)$ we say that γ is <u>positively folded</u> wrt an orientation ϕ if the following holds:

$\forall i$ where γ is folded at i the half-space determined by the hyperplane $H_i := \mathrm{span}(p_i)$ that contains $c_i = c_{i-1}$ is assigned $+1$ by ϕ, i.e. has a <u>positive fold</u>.

<u>Ex:</u>



○ negative fold
○ positive fold

γ is pos. folded wrt c

τ is not pos. folded wrt c

Positively folded galleries arise naturally from retractions in buildings!